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This Journal is dedicated to the following aims:

1. Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values.
2. To supply an additional medium for the publication of expository mathematical articles.
3. To promote more scientific methods of teaching mathematics.
4. To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

Philosophy and Mathematics

More interesting than the fact that many great mathematicians have been accounted also eminent philosophers is, perhaps, the inquiry into the reason.

Is it wrong to ask why Pythagoras, whose principle of proportion is familiar to every school boy, also ventured beyond mere number into metaphysical and ethical speculations? Can we find some relationship between Democritus' eudæmonism and his atomistic theory which influenced the scientific thought of 23 centuries? What is the relation between Euclidean philosophic thinking and the mathematical theses established by the man? What compelled a Plato to have such high regard for number, philosopher though he was primarily? Why did not Kepler, Galileo, and even a Newton, rest content with mathematics and physics, but occupy themselves with metaphysics? Is it fortuitous that the author of the *cogito ergo sum*, Descartes, also became the father of modern mathematics? How can we explain the miscegenation of geometrical and philosophical methods in Spinoza? Are not Leibniz' concepts of continuity and the infinitesimal intrinsically bound up with the doctrine of monads and a pre-established harmony? Would Kant's *Critiques* be as keen if void of mathematical analysis? Why do some famous moderns aspire to mathematical as well as philosophical fame, to name only Bertrand Russell and Professor Whitehead?

A full answer to these questions in their interrelation would constitute a philosophy in itself. But hints might well start our thoughts rambling towards reasons.

Relativists and absolutists both look consciously or unconsciously to a synthetic unity of mathematical and philosophical conception. Philosophy and mathematics go deeper than mere language and psychology. Postulate, let us say, one Principle, raise it to Being in metaphysics, or the unitary Deity in religion, and you immediately invite a quantitative interpretation of the universe. Expound a dualism, and the sheer potency involved in such a world view will oblige you to study dynamics.

Go through the terminology of philosophy and mark how purely mathematical conceptions abound, from monism to pluralism.

"Whole", "part", and the triangular disposition of thesis, antithesis and synthesis, would also suggest a subtle geometry. And who can distinguish the logico-metaphysical from the mathematic in such words used commonly by philosophers and mathematicians, as "time", "space", "relation", "equality", "homogeneity", "class", "quantity", "manifold", "representation", and a host of other *termini technici?* What else is syllogization but a concealed, symbolic logic, a transfigured mathematics?

Yet, we are dwelling on the surface of things. It is not accidental that the Hindu, so highly gifted metaphysically, invented the notion of zero and placed us all under obligation. The close relationship between the speculative and mathematical-physical phases of the Renaissance needs but this brief allusion. It was Oswald Spengler who said: "Every philosophy up till now grew up in intimate connection with a mathematical theory belonging to it." Why? Because number ultimately embodies the meaning of the world as nature. Symbol is either the cipher or the spoken or written word. As such, these "sensible" qualities do not reach the essence of what is intended or meant. The real significance of any philosophy and any mathematics lies beyond the mere formulation of either in a truth or some common ground of reference, at least.

It is from that font that mathematics and philosophy draw liberally, the one operationally, the other speculatively, but neither without the discipline of logic, while the tonicity depends entirely on primary assumptions.

Thus, mathematics and philosophy are identical twins of the mind, and like these have a similar fate.

Mathematics, the language of exact science, and philosophy, the language of adequate description, together constitute integral factors in any modern conception of the universe. The formula, expressed in numerical relationships, is as important as the interpretation. For, while the former teaches us relations on the factual level, the latter endeavors to interrelate the formula and expressions with the world of meaning and values. A purely speculative value, dictated perhaps by emotion, would soon dissociate itself from tangible, verifiable reality. A mathematical formula, born in complete detachment and objectivity, if remaining so, lacks human warmth. There is a common give and take between mathematics, the great unifier of all sciences, on the one

hand, and philosphy the great unifier of all humanistic branches, on the other.

Just as philosphy synthesizes our world image and lends coherence to personal and suprapersonal factors in individual and social life, so mathematics links the universe of law and order with the amenities of the arts in which form, rhythm, and proportion are of fundamental concern. Hence, in the great cultural endeavor to educate whole persons, we can dispense as little with number which upholds and buttresses the technical achievements of the age and subtly influences the plastic and musical arts, as with philosophy which reconciles the thoughts arrived at by rationalization and the ideas that seem like free gifts of a higher genius.

Do we still ask why mathematics and philosophy are wedded in the mind of some of our great ones? There is an inner necessity which compels and draws the two toward each other. Only a mind cynically oblivious to the wonders revealed by a mathematical analysis of physical and astronomical reality, or completely immune to the grandeur of deep thoughts will smile condolingly upon those ancients who regarded these relationships as mysteries and celebrated them in holy rites. Must wonder be a lost virtue? In this scientific age it may still be permissible, as some of our philosopher-mathematicians have proven, to supplement the how explained by scientists by the why of a competent philosophical enquiry.

And thus, while in the search they separate, mathematics and philosophy will ever remain allies in the quest for truth.

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A Series of Line Involutions in S_3 Defined as Point Transformations of a V_4^2 in S_5 Into Itself

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Introduction: The mapping relation between the lines of S_3 and the points of a certain hyperquadric V_4^2 in S_5 is well known, (1),(2),(3)*, and has been of considerable use in the study of the line geometry of three dimensions. For example every line transformation in S_3 is equivalent to a point transformation in S_5 which leaves V_4^2 invariant; and conversely, every transformation of V_4^2 into itself defines a line transformation in S_3. In the present paper a number of involutorial transformations of V_4^2 into itself are defined, and interpreted as line involutions in three dimensions. The series includes the four linear line involutions in S_3, as noted by Klein (4), and several involutions of higher order obtained by Clarkson (5),(6), and by Williams (7).

I. *Definitions of the Involutions; Special Linear Cases:* Consider the following sets of independent linear subspaces of S_5:

1) a point,
2) a line and an S_3,
3) two planes,
4) a plane and two S_3's
5) four S_3's.

Through a general point of S_5 passes a unique line intersecting the subspaces in any one set, and moreover these are the only sets of linear subspaces of S_5 with this property. In particular, with any point P of V_4^2 can be associated the second point P' in which the transversal of a given system which passes through P meets V_4^2. We thus have five involutions which, though they are not defined for all of S_5, do transform V_4^2 into itself, and are therefore equivalent to line involutions in S_3.

Case 1 can be dismissed with a word. The transformation in S_5 is equivalent to the central homology having the given point as center and the polar S_4 of the point with respect to V_4^2 as axis; and in three

*Numbers in parentheses refer to the bibliography at the end of the paper.

dimensions is the familiar transformation associated with a linear complex, (the complex represented in S_5 by the S_4 which is the polar of the given point as to V_4^2).

If the line and the S_3 in case 2 are polars as to V_4^2 the transformation in S_5 is equivalent to the skew involution determined by the given line and the given S_3; and in three dimensions is the linear line involution associated with the involution determined by two skew lines, (the lines represented in S_5 by the two points in which the given line meets V_4^2).

If the two planes in case 3 are polars as to V_4^2 the transformation in S_5 is equivalent to the skew involution determined by the two given planes; and in three dimensions is the linear line involution associated with the polarity defined by a quadric surface, (the quadric which carries the reguli represented in S_5 by the conics in which the given planes meet V_4^2).

If the planes in case 3 are two independent planes lying entirely on V_4^2, the transversal which passes through a point P on V_4^2 lies entirely on V_4^2, and the image point P' cannot be determined as an intersection with V_4^2. However if P' be defined as the harmonic conjugate of P with respect to the points in which the transversal meets the two planes, we have a transformation which in three dimensions is the linear line involution associated with a central homology, (the center being the vertex of the bundle of lines represented in S_5 by one of the planes on V_4^2, the axis being the plane of the field of lines represented by the other plane on V_4^2).

An interesting combination of these special cases arises when we consider all possible transformations which can be derived from the subspaces determined by a set of six points, self-polar as to V_4^2, by taking the points one, two, and three at a time. There are six transformations of type one, fifteen of the special type of case 2, and ten of the first special type of case 3. These, with the identity, form a group of order thirty-two leaving V_4^2 invariant. In three dimensions there is then a group of linear line involutions consisting of the identity, the transformations defined by six complexes in involution, the fifteen transformations defined by the harmonic homologies having as axes the directrices of the congruences in which the complexes intersect by twos, and the ten polarities defined by the quadric surfaces carrying the reguli in which the complexes intersect by threes.

These results are elaborated in (3),(4),(8),(9). They have been summarized here only for the sake of completeness.

II. *Case 2, General Discussion:* To determine the order of the involution defined by the transversals of a line λ and an S_3, Σ, in

general position in S_5, consider any line r on V_4^2. The transversals passing through the respective points of r set up a 1:1 correspondence between the points of r and the points of λ, and hence form a quadric surface which meets V_4^2 in a C_4 consisting of r and a C_3 of which r is a secant. This C_3 is the image of r, and the order of the involution is thus three. The invariant elements are the points common to V_4^2 and another hyperquadric, since r, which was a perfectly general line of V_4^2, contains two points which coincide with their images, (the two points in which r meets its image C_3).

The singular elements include the points A and B in which λ meets V_4^2, the points of the quadric in which Σ intersects V_4^2, and the points of two quadric cones whose generators are the lines of V_4^2 which pass through A and B respectively and intersect Σ. A and B are transformed respectively into the V_3^2's in which the S_4's determined by Σ and A and B in turn meet V_4^2. Any point on the quadric in Σ is transformed into the conic on V_4^2 which passes through the point and through A and B. Any point on either of the quadric cones is transformed into the entire generator on which it lies.

In three dimensions the transformation can be interpreted as follows: Given two fixed lines a and b, and a fixed bilinear congruence. Any line determines a quadratic regulus with a and b, and a linear complex with the congruence. The regulus and the complex have two lines in common, the original line and a second line which is the image of the first. A pencil of lines is transformed into a ruled cubic surface containing two lines of the given pencil (since there is a quadratic complex of invariant lines) and containing a and b (since there are two linear complexes of lines which have a and b respectively for images). The transversal of a and b which passes through the vertex of the pencil is a double directrix of the cubic surface; the transversal of a and b which lies in the plane of the pencil is a simple directrix.

A plane field of lines is transformed into a congruence whose characteristics can be determined by the following considerations in S_5. If π is the plane on V_4^2 which represents the lines of the given plane field, the order of the image congruence is the number of transversals of π, λ, and Σ which meet a general plane of V_4^2 of the system not containing π, (i. e. a plane of V_4^2 having no point in common with π). The class is the number of transversals which meet a plane of the same system as π at points other than the point common to the two planes. To determine the order, consider the S_4 determined by λ and a general plane σ on V_4^2, not meeting π. This S_4 intersects π in a line, and intersects Σ in a plane. The required transversals must lie in this S_4^2, and must meet λ, the plane σ, and the line and plane of

intersection just noted. But in S_4 two lines and two planes have in general just two transversals, therefore the order of the congruence is two. To determine the class, consider the S_4 determined by π and a general plane ρ of V_4^2 which meets π in a point. This S_4 meets λ in a point, and meets Σ in a plane. The required transversals must lie in this S_4, and must meet π, ρ, and the point and plane of intersection just noted. But in S_4 there is in general a unique line passing through a given point and meeting three given planes, therefore the class of the congruence is one. In a similar fashion the image of a bundle of lines can be shown to be a congruence of order one and class two.

The singular lines are 1) a and b, whose images are the linear complexes which they determine with the congruence; 2) the lines of the congruence, whose images are the quadratic reguli which they determine with a and b; and 3) the lines which lie in pencils determined by a or b and a line of the congruence, which are transformed into the entire pencil in which they lie.

If Σ is specialized to intersect V_4^2 in a pair of planes we have as a particular case a transformation discussed by Clarkson (6), from the following definition: Given a cubic regulus and a plane pencil. A line l meets the cubic surface in three points and the plane of the pencil in one point, thus determining three lines of the regulus and one line of the pencil. These four lines have two transversals, l and a second line l', the image of l. We can demonstrate the equivalence of these definitions as follows:

Given an S_3, Σ, meeting V_4^2 in a pair of planes, (the planes are respectively the images of the lines lying in the plane of the pencil in S_3, and the lines passing through the vertex of the pencil), and a line λ meeting V_4^2 in a pair of points A and B. Consider any cubic curve on V_4^2 with the property that each of its points can be joined to A and B by lines lying entirely on V_4^2. (Such a curve is the representation of a cubic regulus in S_3 having the lines represented by A and B as directrices). Consider also a point P and its image P' lying on a line meeting λ and Σ, as required by the first definition. Now P can be joined to three points Q_1,Q_2,Q_3, on the cubic curve by lines entirely on V_4^2. Moreover the tangent S_4 to V_4^2 at Q_i contains P, A, and B, and hence the point of intersection of λ and the line PP'. Therefore P' lies in the tangent S_4 at Q_i, and so can be joined to Q_i by a line entirely on V_4^2. P can also be joined by a line on V_4^2 to one point Q_4 on the line of intersection of the planes in which Σ intersects V_4^2. The tangent S_4 to V_4^2 at this point contains P and each of the planes in which Σ meets V_4^2. It therefore contains Σ, and hence the point in which PP' meets Σ. Therefore P' lies in the tangent S_4 at Q_4, and so

can be joined to the point of tangency by a line lying on V_4^2. In three dimensions these results mean precisely that the lines represented by P and P' meet the same three lines of the regulus, and the same line of the pencil, as required by the second definition. Conversely it is easy to show that if the lines represented by P and P' meet the same three lines of the regulus and the same line of the pencil, the line joining P and P' in S_5 meets a line and an S_3 determined as noted above.

If λ and Σ are so related that the polar S_4 of one of the points, say A, in which λ meets V_4^2, contains Σ, the transversals through the points common to V_4^2 and this S_4 lie entirely on V_4^2. Since any line on V_4^2 meets this S_4 in one point, the image of any line r of V_4^2 is a composite cubic consisting of a conic, which is the proper image of r, and a second line which is the image of the one singular point on r. When the image of this singular point is rejected the involution becomes quadratic instead of cubic.

If the polar S_4's of both A and B contain Σ, any line r of V_4^2 contains two singular points which are transformed into entire lines. These must be rejected before the transformation becomes linear, as noted in Paragraph I.

III. *Case 3, General Discussion:* The involution defined on V_4^2 by the transversals of two general planes, π and π', of S_5 is equivalent to the following involution in three dimensions. Given two quadratic reguli, (represented in S_5 by the conics in which π and α' meet V_4^2). Any line l determines a bilinear congruence with each regulus, and these congruences have two lines in common, l and a second line l', the image of l.

We need not examine this involution in detail for it has been studied from a different definition by Clarkson (5), and by Williams (7). These writers considered independently the transformation defined by the transversals of two generators from one regulus of each of two general quadric surfaces. That this definition is equivalent to the one given above in terms of the transversals of two planes π and π' of S_5 can be established as follows:

Let ρ and ρ' be the planes which intersect V_4^2 in the conics representing the reguli employed in the last definition. Let π and π' be the planes which intersect V_4^2 in the conics representing the other reguli on the two quadrics. Then each point of the conic in ρ (or ρ') can be joined to each point of the conic in the associated plane π (or π') by a line entirely on V_4^2, (since each line of one regulus on a quadric meets every line of the other regulus). Consider now a point P and its image P', lying on a line meeting π and π', as required by the first definition. The point P can be joined to exactly two points on the conic in ρ, and

to exactly two points on the conic in ρ', by lines entirely on V_4^2. Moreover the tangent S_4 to V_4^2 at any one of these points Q_i, contains P and the entire associated plane, π or π'. It therefore contains the point in which the line PP' meets the associated plane, and hence the line PP' itself. Thus P' lies in the tangent S_4 at Q_i, and so can be joined to Q_i by a line entirely on V_4^2. In three dimensions these results imply that the lines represented by P and P' meet the same two lines of each regulus, as required by the second definition. Conversely it is easy to show that if the lines represented by P and P' meet the same two lines of each regulus, the line joining P and P' in S_5 meets the planes of the conics representing the complementary reguli.

For the sake of completeness we note that the order of this involution *is* three, and that there is a quadratic complex of invariant lines. The singular elements in the involution on V_4^2 are the points of the two conics in which π and π' meet V_4^2, which are transformed into quadric surfaces; and the points of a ruled surface of order eight* consisting of those transversals of π and π' which lie entirely on V_4^2. These points are transformed into the entire generators on which they lie.

If π and π' are polars as to V_4^2, any line r on V_4^2 contains two points which are transformed into the entire transversals on which they lie (the transversals through these two points lying entirely on V_4^2). These lines must be rejected before the transformation becomes linear, as noted in Paragraph I.

IV. *Case 4, General Discussion:* To determine the order of the involution defined by the transversals of a general plane π, and two S_3's, Σ and Σ' of S_5, consider the transversals which meet a general line r on V_4^2, a general line in π, and Σ. These form a quadratic regulus, (as noted in Paragraph II) two lines of which meet Σ'. Hence on a general line in π there are two points through which pass transversals of r, Σ and Σ'. The totality of these points in π is thus a conic K, whose points are in 1:1 correspondence with the points of r. The transversals through corresponding points of r and K form a cubic surface which meets V_4^2 in a C_6 consisting of r and a C_5 which is the image of r. The involution is thus of order five.

The invariant points are the points of contact of the tangent lines to V_4^2 which meet π, Σ, and Σ'. Every line r on V_4^2 carries three invariant points, for on r there is a 1:2 correspondence in which to a point P corresponds the two points from which transversals can be drawn to the points where the tangent S_4 at P meets K. A coincidence in this correspondence implies a transversal of π, Σ, and Σ', which is

*This order is incorrectly given as four in (5). The correct value is implied in (7), where the singular surface is obtained as the intersection of three hyperquadrics.

also tangent to V_4^2 at a point of r. Since a 1:2 correspondence on a line has three coincidences, we have our result.

The singular elements are the points of the conic in which π meets V_4^2, the points of the two quadrics in which Σ and Σ' meet V_4^2, and the points of a ruled surface consisting of those transversals of π, Σ, and Σ', which lie entirely on V_4^2. The order of this surface can be determined as follows: Consider any plane in Σ. The lines of V_4^2 which meet this plane and π, form a ruled surface of order eight (as noted in Paragraph III). Eight of these lines meet Σ', so that there is a C_8 in Σ through each point of which passes a unique line of V_4^2 meeting π, and Σ'. (If there were two transversals through a point of the C_8, either π and Σ' would necessarily have a line in common, or the polar line of Σ would necessarily meet π, both of which are specializations contrary to hypothesis.) However through each point of the conic in which π intersects V_4^2 pass two transversals, for the lines through a point of this conic which lie on V_4^2 and meet Σ, form a quadric cone, two generators of which meet Σ'. Thus the points of the conic in π, and the points of the C_8 are in 2:1 correspondence, hence the lines joining corresponding points form a ruled surface of order twelve.

The image of a point on the conic in π is the quadric surface in which the S_3's determined by the point and Σ and Σ' in turn, meet V_4^2. The image of a point on the quadric surface in Σ (or Σ') is the conic common to V_4^2, the S_4 which the point determines with Σ' (or Σ), and the S_3 which the point determines with π. The image of a point on the ruled surface is the entire generator on which it lies.

In three dimensions we have the involution defined by a quadratic regulus and two bilinear congruences. A line l determines a bilinear congruence with the regulus, and two linear complexes with the congruences. These intersect in two lines, l and a second line l', the image of l. The order of the involution is five, and there is a cubic complex of invariant lines, so that a pencil is transformed into a ruled quintic surface containing three lines of the given pencil. An argument similar to that employed in Paragraph II shows that the image of a plane field of lines is a (4,3) congruence, and that the image of a bundle of lines is a (3,4) congruence. The singular elements are the lines of the regulus, the lines of the two congruences, and the lines which lie in pencils containing a line of the regulus and a line of each congruence.

If the S_3's employed in this definition are specialized to intersect V_4^2 in quadrics consisting of a pair of plane, we have the involution defined by the transversals of two lines of a quadratic regulus and one line from each of two general plane pencils. The equivalence of these

two definitions can be demonstrated without difficulty by an argument similar to that employed in Paragraphs I and II. For the details of this special case see (5).

V. *Case 5, General Discussion:* We determine the order of the involution defined by the transversals of four S_3's, Σ, Σ', Σ'', Σ''', in general position in S_5, as follows: The lines which meet Σ', Σ'', a general plane in Σ, and a general line r on V_4^2, meet the plane in Σ in a conic, hence the lines which meet r, Σ, Σ', and Σ'', meet Σ in a quadric surface, which evidently contains the line of intersection of Σ and Σ'. Similarly there is a quadric in Σ through the points of which pass transversals of r, Σ, Σ', and Σ'''; and this quadric also contains the line of intersection of Σ and Σ'. These two surfaces intersect in a C_4 consisting of this line and a residual C_3. Now the two transversals, one of r, Σ, Σ', and Σ'', one of r, Σ, Σ', and Σ''', which pass through the points of the line of intersection of Σ and Σ', will not in general coincide. However the two transversals which pass through a point of C_3 must coincide, else r, which was a perfectly general line on V_4^2, would meet Σ'. The unique transversals through corresponding points of r and C_3 form a quartic regulus, which intersects V_4^2 in a C_3 consisting of r and a C_8 which is the image of r. The involution is thus of order seven.

The invariants points are the points of contact of tangents to V_4^2 which meet Σ, Σ', Σ'', and Σ'''. As before (Paragraph IV) we consider the correspondence on r, in which to a point P corresponds the three points from which transversals can be drawn to the three points in which the tangent S_4 at P meets C_3. A coincidence in this correspondence implies a transversal of Σ, Σ', Σ'', and Σ''', which is also tangent to V_4^2 at a point of r. Since a 1:3 crorrespondence on a straight line has four coincidence, we conclude that every line r, on V_4^2 carries four invariants points.

The singular points are the points of the four quadrics in which Σ, Σ', Σ'', and Σ''' meet V_4^2, and the points of a ruled surface of order sixteen consisting of those transversals of the four S_3's which lie entirely on V_4^2. A point on the quadric in one of the S_3's is transformed into the conic cut from V_4^2 by the three S_4's which the point determines with the other S_4's. A point on the ruled surface is transformed into the entire generator on which it lies.

In three dimensions we have the involution defined by four bilinear congruences. A line l determines a linear complex with each congruence. These four complexes intersect in two lines, l and a second line l', the image of l. The involution is of order seven with an invariant complex of order four, so that a pencil of lines is transformed into a

ruled surface of order seven containing four lines of the given pencil. A plane field of lines is transformed into a (6,5) congruence, and a bundle of lines is transformed into a (5,6) congruence. The singular lines are the lines of the four fundamental congruences, and the lines which lie in pencils containing a line of each congruence.

If the four S_3's are specialized to intersect V_4^2 in a pair of planes we have as a special case the involution studied by Clarkson (5), from the following definition. Given four arbitrary plane pencils. Any line l meets the plane of each pencil in a point, and so determine a line of each pencil. The second transversal of these four lines is l', the image of l. The methods employed in Paragraphs II and III will immediately show the equivalence of these definitions.

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"We have heard much about the poetry of Mathematics but very little of it has yet been sung. The ancients had a juster notion of its poetic value than we. The most distinct and beautiful statement of any truth must take at last the mathematical form. We might so simplify the rules of moral philosophy as well as of arithmetic that one formula would express them both."—From Henry David Thoreau's "*A Week on the Concord and Merrimac Rivers.*"

A Note on the Rational Plane Cubic Curve*

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1. *Introduction.* It is generally known that if the rational plane cubic is expressed parametrically by

$$\rho x_i = (a_i t/s)^3 \equiv a_i t^3 + b_i t^2 s + c_i t s^2 + d_i s^3, \quad i = 0,1,2, \tag{1}$$

that the curve is either degenerate or proper according as the fundamental involution

$$I_2^3{}_{,1} = |a_0 b_1 c_2| t^3 + 3|a_0 b_1 d_2| t^2 s + 3|a_0 c_1 d_2| t s^2 + |b_0 c_1 d_2| s^3 \tag{2}$$

is or is not a perfect cube. Also, the conditions, which determine each type of the proper curve, are known; however, there is no complete method for distinguishing the types of the degenerate curve. This paper establishes such a method.

We shall represent the curve parametrically, although there is a *one to one* relationship between the values of the parameter and the points of only a part of the curve, if it is degenerate.

2. *The degenerate rational plane cubic curve.* Let equations (1) represent any rational plane cubic curve. Then the generally accepted sufficient conditions for (1) to represent a degenerate curve of any type are

$$\frac{|a_0\, b_1\, c_2|}{|a_0\, b_1\, d_2|} = \frac{|a_0\, b_1\, d_2|}{|a_0\, c_1\, d_2|} = \frac{|a_0\, c_1\, d_3|}{|b_0\, c_1\, d_2|}, \tag{3}$$

or the equivalent, where ratios are not used.

However, for some types of degeneracy several of these determinants vanish and hence the conditions become indeterminate, or $I_2^3{}_{,1}$ may even reduce to Kt^3. Moreover there are cases when the fundamental involution (2) vanishes identically and hence again (3) are indeterminate.

First, let us assume conditions (3) to be determinate. This imposes that $I_2^3{}_{,1}$ be the cube of a linear factor, say $gt+hs$, where $gh \neq 0$.

*Presented to the American Mathematical Society September 9, 1937.

Then any binary form apolar to (2) must contain this linear factor. Hence all cubics apolar to (2) are of the form

$$(gt+hs)(mt^2+2nts+ps^2),$$

where m, n, and p are arbitrary.

Using this fact, we are able to write equations (1) as follows:

$$\rho x_i=(Lt+s)[a_it^2+L(c_i-Ld_i)ts+Ld_is^2], \quad i=0,1,2, \tag{4}$$

where

$$L=g/h=\frac{|a_0\ b_1\ c_2|}{|a_0\ b_1\ d_2|} \neq 0,$$

and *the quadratics are not restricted.*

The parameter value $t/s=-1/L$ makes that point arbitrary and we shall, throughout this paper, interpret such a representation as (4) to be the conic defined by the remaining factors and any line* on the point $t/s=-1/L$ of that conic.

That is, *the* $(a_it/s)^3$ of (1), *under determinate conditions* (3) *must have a common factor if* (1) *represents a conic and a line.*

Now let the three quadratics in (4) represent a degenerate conic. If they are in a parabolic involution,† they must be apolar to the square of a linear form. Thus, as above, they must have a common linear factor, the linear form. The converse of this is also true.

If the square of the linear form is $(\alpha t+\beta s)^2$, we are able to write (4) as

$$\rho x_i=(Lt+s)(\alpha t+\beta s)(\beta a_it+\alpha Ld_is), \quad i=0,1,2. \tag{5}$$

This may be considered as representing *three distinct lines*, the line defined by the last factor in each x_i, and any line on each of the points $t/s=-1/L$, $-\beta/\alpha$ of that line. The three lines are on a point if $1/L=\beta/\alpha$.

But if equations (1) can be written in the form (5), then conditions (3) and hence L, as defined above, become indeterminate. Also the fundamental involution no longer exists as a definite cubic but is replaced by the pencil of cubics.

$$\lambda(Lt+s)^3+\mu(\alpha t+\beta s)^3, \tag{6}$$

formed from the common factors of equations (5).

*H. I. Thomsen, American Journal of Mathematics, *The Osculants of Plane Rational Quartic Curves*, 32, 1910, p. 207.

†R. M. Winger, *Projective Geometry*, 1923, p. 157.

However, L may now be found as the common root of the equations

$$d_iL^3-c_iL^2+b_iL-a_i=0, \tag{7}$$

which arise in expressing (1) in the form (4), due to conditions (3). Furthermore, for three lines on a point, L is a double root of equations (7) and may be readily found from their derived equations. Hence (1) *represents three dependent lines if* $b_il^2+2c_ils+3d_is^2=0$ *have a common root which is a double root of* $(a_il/s)^3=0$, *and if conditions* (3) *are indeterminate.*

The remaining possibility is that the quadratics in (4) be in a proper involution. This makes the $(a_il/s)^3$ linearly related, just as in (5), and the conic becomes a repeated line, since to each point corresponds two parameter values.* This is seen to be true since (4) may now be written in the form

$$\rho x_0=-k(Ll+s)(\gamma_il/s)^2, \quad \rho x_1=-k(Ll+s)(\delta_il/s)^2, \tag{8}$$
$$\rho x_2=[m(\gamma_il/s)^2+n(\delta_il/s)^2](Ll+s), \quad i=0,1,2.$$

Whence $mx_0+nx_1+kx_2=0$, and for each point (x) of (8) there are two parameter values, or the line has a double spread of the parameter system. Again the fundamental involution becomes a pencil of cubics. Under this condition, equations (4) give this pencil as

$$\lambda(Ll+s)^3+\mu(3l^2s+3mls^2+ns^3), \tag{9}$$

where l, m, and n are formed from the coefficients of the quadratic factors. They are

$$l=L^2\begin{vmatrix} a_0 & c_0-Ld_0 \\ a_1 & c_1-Ld_1 \end{vmatrix},$$

$$m=L\left\{2\begin{vmatrix} a_0 & c_0-Ld_0 \\ a_1 & c_1-Ld_1 \end{vmatrix}+\begin{vmatrix} a_0 & 2Ld_0-c_0 \\ a_1 & 2Ld_1-c_1 \end{vmatrix}\right\},$$

$$n=3\begin{vmatrix} a_0 & c_0-Ld_0 \\ a_1 & c_1-Ld_1 \end{vmatrix}+3\begin{vmatrix} a_0 & 2Ld_0-c_0 \\ a_1 & 2Ld_1-c_1 \end{vmatrix}+\begin{vmatrix} c_0-3Ld_0 & L^3d_0-a_0 \\ c_1-3Ld_1 & L^3d_1-a_1 \end{vmatrix}.$$

Hence, *if the quadratics in* (4) *are in an involution, whether proper or not, equations* (1) *represent three lines, and the fundamental involution, as given by* (2), *disappears and may be replaced by a pencil of cubics.*

*Brill, Mathematische Annalen, *Concerning the rational plane quartic curve*, 12, 1875, p. 102; Thomsen, l. c.

Finally, if the $(a_it/s)^3$ of (1) may be changed to linear forms by a transformation of $t'/s' = (\alpha t/s)^3/(\beta t/s)^3$, then *the curve is the cube of a line*, as the x_i are linearly related; also, to each point (x) will correspond three parameter values.*

In this case also, the generally accepted fundamental involution vanishes identically, and the degeneracy conditions become indeterminate, and we again have a pencil of apolar cubics. As an example of the cube of a line, consider the cubic given by

$$\rho x_i = a_i t^3 + b_i s^3, \quad i = 0,1,2. \tag{10}$$

Here $t' = t^3$, $s' = s^3$ reduces the $(a_it/s)^3$ to linear forms, and they have the apolar pencil of cubics

$$\lambda t^2 s + \mu t s^2.$$

*Thomsen, l. c.

TICKLERS

From an examination paper:

"$\sin \Theta = \dfrac{\text{opposite side}}{\text{hypothesis}}$".

Extract from an examination paper:

"A radian is an angle at the center of the circle when the subtrahended arc is equal to the radius."

The following "proof" of the Remainder Theorem has been circulated widely enough that its origin is pretty well obscured.

To prove that the remainder obtained on dividing the polynomial $f(x)$ by $x-r$ is $f(r)$.

$$\begin{array}{r|l} x-r & f(x) \\ \hline & f(x) - f(r)\dagger \\ \hline & f(r) \end{array} \quad \Big| \; f^*$$

* x into $f(x)$ is f.
† f times $x-r$ is $f(x)-f(r)$.

In one question of an examination in solid analytics it was necessary to find the direction numbers of a line passing through (say) the points (2,4,5) and (6,1,2). One student (?) found the direction numbers to be (3,6,7). The instructor was much perplexed in reading the paper to know how the student had arrived at these numbers, until he found the computation in the scratch work:

$$\begin{array}{r} 612 \\ 245 \\ \hline 367 \end{array}$$

On the Rational Plane Quartic Curve with Triple Point

By H. E. Arnold and A. R. Parente
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The rational plane quartic curve can be represented by the three binary quartics

$$x_i = a_i t^4 + b_i t^3 \tau + c_i t^2 \tau^2 + d_i t \tau^3 + e_i \tau^4, \qquad (i = 1,2,3). \tag{1}$$

If an arbitrary quartic

$$\alpha t^4 + 4\beta t^3 \tau + 6\gamma t^2 \tau^2 + 4\delta t \tau^3 \epsilon \tau^4, \tag{2}$$

where the binomial coefficients are used for convenience, is assumed to be apolar to the quartics (1), then three linear conditions*

$$\alpha e_i - \beta d_i + \gamma c_i - \delta b_i + \epsilon a_i = 0, \qquad (i = 1,2,3), \tag{3}$$

are imposed on the coefficients of (2). Since one coefficient in (2) may therefore be chosen arbitrarily, we have an involution of quartics, which we write symbolically

$$k(\beta t)^4 + k'(\gamma t)^4, \tag{4}$$

apolar to x_i and hence apolar to any line section of the curve. Conversely, any quartic apolar to (4) represents a line section. The involution (4) has been called the Fundamental Involution of the quartic curve, and the conditions for various special types of the curve have been expressed in terms of the combinants of this involution.†

But the pencil of lines through any point A of the plane also gives rise to an involution of quartics, for by substituting the x's from (1) in

$$\rho(\lambda x) + \sigma(lx), \tag{5}$$

where $(\lambda x) = \lambda x_1 + \mu x_2 + \nu x_3 = 0$ and $(lx) = lx_1 + mx_2 + nx_3 = 0$ are any two lines through A, we have an involution

$$\rho(\beta' t)^4 + \sigma(\gamma' t)^4. \tag{6}$$

*Winger, *Projective Geometry*, Chapter VIII.

†Rowe, *Covariants and Invariants of the Rational Plane Quartic*, Transactions of the American Mathematical Society, Vol. 12 (1911) pp. 301-304.

A condition imposed on the combinants of (6) means that the point A is restricted to some locus, and a number of these covariant loci have been obtained.*

Now an involution of quartics possesses a unique apolar sextic, since six conditions are imposed on the coefficients of the sextc by making it apolar to two independent quartics ot the involution. Suppose the sextic f^6 is apolar to the involution (6). By means of f^6 a relation between (4) and (6) has been found for the general case of a curve with three double points.† It is the purpose of this note to discuss the special case of the curve with a triple point.

When a conic is passed through the point A, the double points B, C, D, and any other point E on the curve, another point F on the curve is determined. The second polar of f^6 with respect to the parameters of E and F gives a quartic which is a member of the fundamental involution (4), since it can be shown to be apolar to any line section of the curve. If we let the double points coincide to form a triple point, say at B, the conic must have six intersections with the quartic at that point. Hence the conic degenerates to two lines through B, one arbitrary and the other passing through A. Let E' be the fourth intersection of the line AB with the curve, and F' the point, other than the triple point, at which the arbitrary line through B cuts the curve. Then the second polar of f^6 with respect to the parameter of E' and any arbitrary parameter gives a member of the fundamental involution, or, the fundamental involution is just the set of first polars of the quintic obtained by polarizing f^6 with respect to the parameter of E'. It is well known that when the curve has a triple point the fundamental involution is the system of first polars of a quintic, and that the parameters of the triple point are the roots of the canonizant of this quintic.‡ We have shown here how to obtain this fundamental quintic from the pencil of line sections (6).

The analytic proof of the above relation between (4) and (6) affords an interesting exercise in the use of polar and apolar forms. Let the curve be given by

$$(7)\qquad \begin{cases} x_1 = t^3\tau + t^2\tau^2 \\ x_2 = t^2\tau^2 + t\tau^3 \\ x_3 = a_3t^4 + b_3t^3\tau + c_3t^2\tau^2 + d_3t\tau^3 + e_3\tau^4, \qquad (a_3,\ e_3 \neq 0), \end{cases}$$

where the triple point is at the vertex (0,0,1) and has the homogeneous parameters (0,1), (1,0), and (−1,1). The pencil of lines (5) may,

*Rowe, *loc. cit.*, pp. 298-301.

†Arnold, *Concerning the Rational Curves* R_3^5(II) *and* R_2^4, American Journal of Mathematics, Vol. 52 (1930), pp. 605, 606.

‡Landry, *A Geometrical Application of Binary Syzygies*, Transactions of the American Mathematical Society, Vol. 10 (1909), pp. 96, 97.

without loss of generality, be simplified by letting ν and either l or m, say m, be zero. The involution (6) then becomes

$$\rho\{\lambda t^3\gamma+(\lambda+\mu)t^2\tau^2+\mu t\tau^3\}$$

$$+\sigma\{na_3t^4+(l+nb_3)t_3\tau+(l+nc_3)t^2\tau^2+nd_3+t\tau^3+ne_3\tau^4\}, \tag{8}$$

and the parameters of E' are $t=-\mu$, $\tau=\lambda$. It can readily be verified that the sextic f^6 apolar to (8) is

$$\begin{aligned}
&a[\mu^3a^2-(\lambda\mu^2+\mu^3)ab+(\lambda^2\mu+\mu^3)ac-(\lambda^3+\mu^3)ad\\
&\quad-(\lambda^3+\lambda^2\mu+\lambda\mu^2)ae+\lambda\mu^2b^2-(\lambda^2\mu+\lambda\mu^2)bc\\
&\quad+(\lambda^3+\lambda\mu^2)bd+(\lambda^3+\lambda^2\mu)be+\lambda^2\mu c^2\\
&\quad-(\lambda^3+\lambda^2\mu)cd-\lambda^3ce+\lambda^3d^2]t^6\\
&-6ae[(\lambda^3+\lambda^2\mu+\lambda\mu^2+\mu^3)a-(\lambda^3+\lambda^2\mu+\lambda\mu^2)b\\
&\quad+(\lambda^3+\lambda^2\mu)c-\lambda^3d]t^5\tau\\
&-15ae[(\lambda^2\mu+\lambda\mu^2+\mu^3)a-(\lambda^2\mu+\lambda\mu^2)b+\lambda^2\mu c-\lambda^3e]t^4\tau^2\\
&-20ae[(\lambda\mu^2+\mu^3)a-\lambda\mu^2b+\lambda^2\mu d-(\lambda^3+\lambda^2\mu)e]t^3\tau^3\\
&-15ae[\mu^3a-\lambda\mu^2c+(\lambda^2\mu+\lambda\mu^2)d-(\lambda^3+\lambda^2\mu+\lambda\mu^2)e]t^2\tau^4\\
&-6ae[\mu^3b-(\lambda\mu^2+\mu^3)c+(\lambda^2\mu+\lambda\mu^2+\mu^3)d\\
&\quad-(\lambda^3+\lambda^2\mu+\lambda\mu^2+\mu^3)e]t\tau^5\\
&+e[\mu^3ac-(\lambda\mu^2+\mu^3)ad+(\lambda^2\mu+\lambda\mu^2+\mu^3)ae-\mu^3b^2\\
&\quad+(\lambda\mu^2+\mu^3)bc-(\lambda^2\mu+\mu^3)bd+(\lambda^3+\mu^3)be\\
&\quad-\lambda\mu^2c^2+(\lambda^2\mu+\lambda\mu^2)cd-(\lambda^3+\lambda\mu^2)ce\\
&\quad-\lambda^2\mu d^2+(\lambda^3+\lambda^2\mu)de-\lambda^3e^2]\tau^6,
\end{aligned} \tag{9}$$

where $a=na_3$, $b=l+nb_3$, $c=l+nc_3$, $d=nd_3$, $e=ne_3$. Polarizing (9) with respect to $(-\mu,\lambda)$ we have, upon dropping a constant factor, the quintic

$$\begin{aligned}
&[(a_3-b_3+c_3-d_3)/e_3]t^5-5t^4\tau-10t^3\tau^2\\
&\qquad-10t^2\tau^3-5t\tau^4-[(b_3-c_3+d_3-e_3)/a_3]\tau^5.
\end{aligned} \tag{10}$$

We note that this quintic is independent, as it should be, of the position of A.

The fundamental involution of the curve is found to be

$$\begin{aligned}
&\kappa\{[(b_3-c_3+d_3)/e_3]t^4+4t^3\tau+6t^2\tau^2+4t\tau^3\}\\
&\qquad+\kappa'\{4t^3\tau+6t^2\tau^2+4t\tau^3+[(b_3-c_3+d_3)/a_3]\tau^4\},
\end{aligned} \tag{11}$$

and the system of first polars of the base quintic (10) can easily be identified with (11).

The cases of curves which have triple points with parameters which are not distinct can be treated in like manner.

A Ruler and Compass Mapping Problem Associated with the Equilateral Triangle

By J. J. L. HINRICHSEN
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The following note concerns a mapping problem in which the images of points may be found in a simple manner by ruler and compass construction. The transformation considered is two-to-one on the interior of a certain region lying within the equilateral triangle. This property does not pertain to the boundary for reasons which become obvious from the ruler and compass construction.*

Consider the equilateral triangle ABC. With A as center and the altitude of the triangle as radius, swing an arc $A'B''$ from the center A' of BC to the point B'' on the side AC. Let S denote the region $A'B''C$ and let P be a point of S. Form a triangle ABP and construct the lines AR_i, $(i=1,2)$ and PG of length

$$\frac{\sqrt{3}}{2}\,\overline{AB}$$

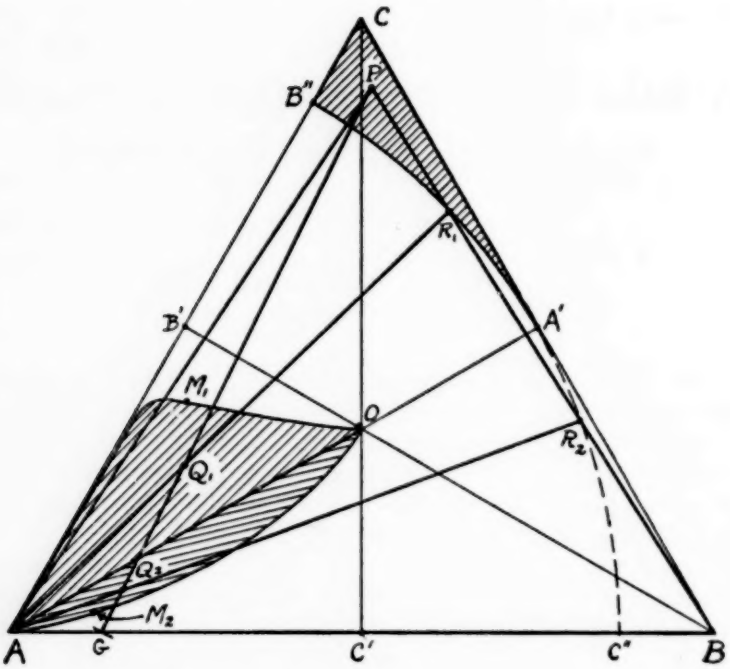
extending from the vertices A and P to the opposite sides BP and AB respectively ($\angle BGP<90°$). If the points of intersection are Q_i, the transformation to be considered is that which carries P into Q_i, $(i=1,2)$.

This transformation may be considered as carrying the vertex P of a triangle ABP with fixed base into the points of intersection of the two line segments PG and AR_i of length

$$\frac{\sqrt{3}}{2}\,\overline{AB}$$

with the condition $\angle BGP<90°$. To obtain an analytical expression for the points Q_i in terms of the point P, let us introduce a coordinate system with origin at B and $x-$axis along AB, and apply the ordinary

*This problem arose in the author's attempt to establish a theorem which he proposed to the Problems Department of the *American Mathematical Monthly* in 1932 and which was proved in 1935 independently by E. H. Cutler and J. K. Peterson. See *American Mathematical Monthly* Volume XLII, pages 178-180.



methods of analytic geometry. If the coordinates of A, P, and Q_i are denoted by

$$\left(-\frac{2}{\sqrt{3}},0\right), \quad (\xi,\eta), \quad (x_i,y_i)$$

respectively, the transformation may be written

$$x=\frac{-2}{\sqrt{3}}+\frac{\left(\xi+\frac{2}{\sqrt{3}}-\sqrt{1-\eta^2}\right)(2\eta^2\pm\xi\sqrt{3\xi^2-\eta^2})}{2\eta^2+2\xi\sqrt{1-\eta^2}\pm(\xi-\sqrt{1-\eta^2})\sqrt{3\xi^2-\eta^2}}$$

$$y=\frac{\eta\left(\xi+\frac{2}{\sqrt{3}}-\sqrt{1-\eta^2}\right)(-2\xi\pm\sqrt{3\xi^2-\eta^2})}{2\eta^2+2\xi\sqrt{1-\eta^2}\pm(\xi-\sqrt{1-\eta^2})\sqrt{3\xi^2-\eta^2}},$$

where the upper signs give x_1, y_1, and the lower signs x_2, y_2.

To find the points Q_i, $(i=1,2)$, corresponding to a given point P, analytically by means of these formulas, would involve considerable computation. The ruler and compass method of determining these

points however is very simple. The interior of the region S is transformed in a one-to-one manner into the interior of each of the regions OAM_iO. The line segment $B''C$ is carried into the curves AM_iO; the point C is carried into the point O; the line segment CA' is carried into the line segment OA. Thus the region $B''CA'$ which is open along the arc $A'B''$ is transformed in a one-to-one manner into each of the regions OAM_iO, $(i=1,2)$ which is open in the point A. Since neither the geometric nor the analytic definitions of the transformation apply to points along the arc $A'B''$, the transformation may be defined to carry the arc $A'B''$ into the point A. This definition is a continuous extension of the definition of the transformation for interior points of the region. The closed region $CA'B''$ is then carried into each of the closed regions OAM_iO, $(i=1,2)$ but no longer in a one-to-one manner.

Let us observe the effect of the transformation upon various simple curves. A segment of any straight line through B will transform into a segment of a straight line through A while a segment of any straight line through C will be carried into a segment of a regular curve joining the points A and O. Any arc of a regular curve joining a point of $A'B''$ to a point of $A'C$, a point of $B''C$ to a point of $A'C$, or a point of $A'B''$ to a point of $B''C$ will be transformed into an arc of a regular curve joining A to a point of AO, a point of AM_iO to a point of AO, or A to a point of AM_iO respectively. Any curve lying in S which meets the arc $A'B''$ n times will be transformed into a curve lying in OAM_iO having n points coincident with A and hence having $(n-1)$ loops.

Finally consider the transformation which carries the regions OAM_1O and OAM_2O into one another. The explicit analytical representations

$$\begin{cases} x_2=\phi_1(x_1,y_1) \\ y_2=\phi_2(x_1,y_1) \end{cases} \quad \text{or} \quad \begin{cases} x_1=\psi_1(x_2,y_2) \\ y_1=\psi_2(x_2,y_2) \end{cases}$$

are difficult to obtain. The ruler and compass method fails in as much as it is impossible to pass from either Q_1 or Q_2 to P. A ruler and compass method of trial and error may however be applied to obtain as accurately as desired the point corresponding to a given point. This latter transformation is one-to-one inclusive of the boundaries and may be considered as a generalization of a reflection in a straight line.

Humanism and History of Mathematics

Edited by
G. WALDO DUNNINGTON

A Second Lesson in the History of Mathematics

By G. A. MILLER
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4. *Differences in attainments.* In order to understand the history of mathematics it is necessary to bear in mind the varied mathematical attainments of people living at the same time. Hence it is impossible to divide this history into completely separated periods of time with respect to the mathematical attainments of all the people. In particular, there have always been people who belonged to the pre-Grecian era as regards their mathematical interests and abilities. This includes not only the young children but also many of maturer years. As every individual has to start from the beginning in acquiring his insight into abstract mathematics, and different intellectual interests attract his attention in an increasing multiplicity of forms, it is only natural that there are now comparatively few who acquire the aptitude which makes mathematical pursuits a pleasure and who enjoy the related philosophical speculations which the ancient Greeks emphasized and which have since then remained an integral part of our subject in various degrees.

What has been commonly embodied in our histories of mathematics relates to comparatively few people living at the periods concerned and frequently had little or no influence on the activities of the rest of the population. Even the evidences of it in the construction of buildings and in the financial management of the affairs of the nations failed to attract wide attention in early times. ***The mathematical Stone Age is still being illustrated by the intellectual attainments of many of our people.*** Even in educated circles it often happens that only the results of certain developments are appropriated but the ability to establish these results is widely lacking. In fact, in many

[EDITOR'S NOTE: This article is a continuation of Professor Miller's article *A First Lesson in the History of Mathematics* in Vol. XIII, No. 6, pp. 272-277 (March, 1939), issue of this journal.]

cases this ability is shamelessly lacking, especially in cases where abilities along other intellectual lines have been cultivated to a conspicuous extent and are widely appreciated.

The history of ancient mathematics is not confined to the study of ancient records of mathematical attainments although these constitute its most important feature. In §1 (volume 13, page 272, 1939) we referred to the word one-half as evidence of the early use of this fraction and as indirectly pointing to the early use of other fractions. It may be desirable to emphasize here the fact that there are good reasons for assuming that the fraction one-half may have been commonly used at least as early as the number 2. There are many evidences which point to the influence of implements of measure on the early development of our number system. In particular, the ancient Egyptian general symbol for a unit fraction is also their symbol for a small measuring implement. It is easy to see that the need of an implement to measure one-half as much might have made itself felt even earlier than the need of one to measure twice as much, since in the latter case the ancients could do the thing over, but to measure one-half the process is not quite so easy.

Possibly the human race did not acquire a knowledge of numbers in exactly the order in which the modern child acquires it. The fact that early number names are connected with the names of implements of measure illustrates the close connection between pure and applied mathematics even in the earliest times of which we have records. Pure and applied mathematics supplemented each other during the entire period covered by the mathematical records. While applied mathematics predominated in the advanced fields of this subject during the eighteenth century, as may be seen from the works of L. Euler (1707-1783), who was probably the most prolific mathematical writer of all times,* there does not seem to have been any long continued modern change as regards the relative emphasis on these two branches of mathematics. It should be emphasized that no clear line of demarcation between pure and applied mathematics exists. The treatment of these subjects is frequently interwoven in the literature of our subject and their names illustrate the fact that many vague terms still appear in this literature.

The fact that the mathematical development of each individual portrays in some respects the history of mathematics as a whole naturally invests this history with unusual interest and gives us many clues in regard thereto. The study of these clues complicates the study

*Euler's *Collected Works* are still in the process of publication, although he died more than 150 years ago.

of the history of our subject since it involves the study of various other subjects, such as the history of languages. There is, however, a great difference between the study of such clues and the study of myths based on unreliable reports, such as the one about J. J. Sylvester's striking a student with a sword-cane.* Unfortunately, many of these still appear in the literature of our subject and their removal constitutes one of the most serious tasks which confronts those who aim to contribute to the history of our subject, since they are apt to encumber real advances. Definite clues on the contrary are apt to lead to explanations of actualities and constitute reasonable incentives to study. They are apt to reveal connections which had not been foreseen and to arouse thought in regard to probabilities.

The mathematical historian frequently desires to associate mathematical discoveries with individuals whenever this is possible. When this is impossible it is natural to seek to associate such discoveries with the people who seem to have been the first to acquire them as their common property and who later transmitted them to other nations. All the pre-Grecian history of mathematics concerns itself only with the latter mathematical attainments. With the ancient Greeks, on the contrary, individuals began to receive credit for their contributions along mathematical lines. While this led to a more satisfactory mathematical history it also introduced new difficulties since mathematical results were frequently due to many and the one who made the finishing contributions thereto is often the only one who is remembered. *Mathematics is, in fact, the gift of the human race and not only the gift of the thousands of individuals of the race whose names are recorded in the mathematical histories.*

It is impossible to determine the number of those who have made important contributions towards the development of our subject. In many cases incomplete contributions have suggested further progress to others. Like seeds in the ground they lost their identity but were duplicated manifoldly in the minds of others who were able to profit by their suggestions. It is usually impossible to trace to their ultimate sources various mathematical ideas since the investigators themselves are usually unable to give due credit to all of those who inspired their work and who in some cases purposely obliterated sources. Hence the tendency to give mathematical credit to individuals, which was inaugurated by the ancient Greeks and has since then gained more and more in popularity, is not free from great historical difficulties even if it is supported by the kindest human feelings. In speaking of the

*A. Macfarlane, *Ten British Mathematicians*. (1916). p. 108. It is difficult to eradicate even such modern myths from the histories of our subject.

discoveries of Eudoxus (about 400 B. C.) with respect to the volume of a pyramid, Archimedes (287-212), who has often been called the greatest mathematician of antiquity, said in his work *On the sphere and cylinder*, near the opening of Book 1, "For these properties also were naturally inherent in the figures all along yet they were in fact unknown to all the many able geometers who lived before Eudoxus, and had not been observed by any one." *The history of mathematics concerns itself primarily with those who first published results.* The date of publication and the date of discovery may differ widely.

Such simple and fundamental ideas as the fact that the sides of similar triangles are proportional can probably never be traced to their sources. The fact that the ancients gave the same name to the totality of some of the similar figures, such as circles and squares, is evidence of the great entiquity of the notion of similarity and prepared the way for the later study of the transformation groups which transform such figures into themselves. Mathematical historians used to credit the ancient Greek mathematician, Thales (about 600 B. C.), with the discovery of the fact that the sides of similar triangles are proportional, but recent discoveries in regard to the mathematics of the ancient Babylonians and the ancient Egyptians prove that this theorem is much older, and that about 2000 B. C. there was much greater mathematical activity than had been suspected by the former mathematical historians. Hence the modern student of our subject should use older historical accounts with great caution and may reasonably expect to find shortcomings therein.*

5. *Decimal system.* The decimal system of numeration and notation is now the most widely used numerical system and seems to have been one of the most favored systems since the earliest records with respect to the use of integers but not with respect to the use of fractions. In fact, none of the ancients are known to have made special use of it in their numerations or notations of fractions with respect to a base. Such common fractions as $^1/_2$ and $^1/_5$, for instance, were not represented by any of the ancients in a form similar to .5 or .2, respectively, but were always represented in a special form when the base 10 was used. When the base 60 was used the situation is entirely different, since the ancient Babylonians represented the given fractions by the numbers 30 and 12 respectively. The history of mathematics therefore exhibits a system of numerical notation to the base 10 which was predominantly used by the ancients with respect to integers and a system to the base 60 which was predominantly used by them when fractions were repre-

*The *Vorlesungen über die Geschichte der Antiken Mathematischen Wissenschaften* by O. Neugebauer (Vol. 1, 1934), now at Brown University, are especially useful.

sented with respect to a base of notation and numeration without a special notation based on that of integers.

Finally the decimal base also won out with respect to fractions but this did not take place until medieval and modern times and then only very gradually. In medieval times there was an approach to decimal fractions when in division and in the extraction of square roots a number of zeros was added to the number before the operation was performed and then the proper adjustment was made in the results. For instance, when it was desired to find $\sqrt{26}$ the approximate $\sqrt{260,000}$ was often found and then this result was divided by 100. This was, however, not yet a fully developed decimal system since the people then dealt directly only with integers and common fractions. The same remark applies to the early trigonometric tables which were computed with respect to circles whose radius was very large, such as 10^5 in some of the work of Regiomontanus (1436-1476), and some larger values for the radius in the works of various other writers in medieval times.

While a number of other bases were used to a comparatively small extent in the representation of integers, the base 60 is the only one which was employed extensively in ancient times for the representation of fractions as well as integers, and hence it occupies a unique position in the history of our subject. This is another element of unusual historical interest with respect to the sexagisemal system of numerical notation and numeration. It is probably related to the fact that 60 is divisible by all the prime numbers which are less than 7 and it may also be related to the fact that in early times numbers were closely connected with implements of weight and measure as was noted above. The widespread use of the decimal system when integers are involved is commonly assumed to be due to the fact that we have ten fingers on our hands, which constitute convenient objects to establish a one to one correspondence between small integral numbers of objects of various kinds.

Very much has been written in regard to the advantages of different bases of systems of numeration and notation. The noted French mathematician B. Pascal (1623-1662) considered already the scientific advantages of number systems having different bases. The first article of the *Bulletin of the New York Mathematical Society* (1891-2), which is the predecessor of the *Bulletin of the American Mathematical Society* (1894-), was devoted to the octonary numeration. Many writers have advocated the adoption of the base 12 but their efforts to secure popular support have thus far not met with much success. It may be noted that the somewhat common statement that *the terms*

eleven and twelve are evidences of the superiority of 12 as a base is not in accord with the facts since eleven means one left over and twelve means two left over, and hence these terms relate to the base 10 but not to the base 12.*

In the use of decimal fractions one usually abandons the field of exact mathematics and enters the field of approximate mathematics. As measurements are necessarily approximations there is naturally a wide field of usefulness for approximate mathematics and it is desirable to become familiar with the degree of approximation which is allowable in various types of mathematical computations. When the data involved are necessarily inexact it is clearly incongruous to insist on exact mathematical computations relating thereto. The history of exact mathematics and the history of approximate mathematics are frequently interwoven and they are very extensive. While the latter avoids many of the difficulties of the former it also introduces new difficulties since it involves the question of permissible errors, which vary with the data under consideration and the purposes for which the results are to be used. This explains why such a great variety of logarithmic tables, involving different numbers of decimal places, are in common use at the present time.

The ancient Egyptian numerical system is decimal but not positional. It employs different symbols for the different powers of 10, at least as far as a million, in early times, but the symbols for one million and for one hundred thousand do not appear in the later writings of the ancient Egyptians, due to the fact that they became subject to foreigners who dominated their cultural activities. This furnishes an interesting example of the fact that the use of large numbers is commonly affected by the cultural attainments of a people. This use has had various motives. In modern times these numbers have been connected with the use of money and with various scientific questions, such as the distances to stars and the quantum theory. *Our Bible contains no number which is as large as one million* but the ancient Greek mathematicians, including Archimedes and Apollonius, considered much larger numbers from a scientific point of view. Even the ancient Babylonians used numbers occasionally which are much larger than a million.

One of the important historical facts which the extensive use of the decimal system illustrates is that the development of mathematics was largely influenced by factors which were not controlled by men.

*Cf. D. E. Smith, *History of Mathematics*, Vol. 1, page 12 (1925). The same error appears in the article on *Arithmetic* in the *Encyclopædia Britannica* (1938). It may be added that this encyclopædia does not include a biography of J. J. Sylvester, but it includes biographies of some minor American mathematicians.

The fact that we have ten fingers seems to be of fundamental importance in the number systems of various countries and it seems to have been more influential in the early development of these systems than any theoretical considerations relating to the advantages inherent in the use of different bases for systems of numeration and notation. *The history of mathematics should be regarded as an intrinsic element of the history of the intellectual development of the human race* and it should not be confined to the consideration of the mathematical facts relating thereto. Thinking people naturally become interested in mathematical considerations since the latter are the embodiment of the purest and most far-reaching relations, both practical and theoretical.

6. *Generalizations.** While it would be difficult to prove that mathematics has become relatively more abstract since the time of the oldest extant literary mathematical records one is confronted in many directions with evidences of increasing generalizations since the time when these early records were made. For instance, the number concept has been greatly generalized by the introduction of complex numbers during medieval and modern times and the use of more than three dimensions in geometry has greatly enlarged the scope of geometric considerations. Abstraction itself is a kind of generalization in a certain direction. The general concept which is now commonly denoted by the symbol $f(x)$ has no parallel in ancient mathematics. The comparatively modern terms of function and group are not duplicated in meaning by any ancient mathematical terms and are evidences of a growing tendency towards generalizations in modern mathematics, which inspired further progress.

An important feature of generalizations in mathematics is that they sometimes seem to have forced themselves on the mathematicians and had much more fruitful consequences than were foreseen by those who first employed them. Wise mathematical forward steps were frequently taken by those who apparently failed to foresee their profound advantages. This is illustrated by the introduction of the negative numbers which were used occasionally long before a satisfactory theory relating thereto had been published. For centuries people used these numbers in accord with certain rules which they had found to lead to correct results in many cases in calculation before it was proved that they can be thus used legitimately. Hence one of the greatest and most useful steps towards the generalization of mathematics had its origin in the persistent use of negative numbers by those who had not fully established the legality of this use. Insecure forward steps have frequently inspired others which became secure.

*G. A. Miller, *Collected Works*, Vol. 2, p. 495 (1938).

As an illustration of the use of negative numbers by those who failed to understand their theory, we may refer to a statement in the algebra of C. Clavius (1608) to the effect that it is impossible for the human beings to understand the rules of multiplication with negative numbers but their correctness cannot be doubted since it has been illustrated by many examples. The history of negative numbers is long and represents one of the most inspiring triumphs of the human intellect in its efforts to simplify fundamental ideas. Even the mathematics of the eighteenth century was beclouded in many places by a lack of a clear understanding of the use of negative and complex numbers, and the mathematics of all the earlier centuries suffers greatly as a result of a lack of knowledge along this line. This is a very profound and primary fact of the history of mathematics which should be continually borne in mind in the study of this history as far as it relates to developments preceding the nineteenth century. Its implications are both profound and extensive. The European mathematicians were the first successful explorers of this field at about the beginning of the nineteenth century.

One of the serious obstacles which presented themselves to those who first considered the use of negative numbers is the fact that this use makes it necessary to abandon the view that the ratio of a larger number to a smaller number must necessarily exceed unity since the ratio of -3 to -4 is $^3/_4$, while -3 is larger than -4. In dealing with positive numbers only the ratio of a larger number to a smaller number always exceeds unity and this is the main reason why we regard one of these two positive numbers larger than the other. It is easy to see that the abandonment of such a fundamental view presented a serious reason to many why negative numbers should not be regarded as actual numbers. The adoption of negative numbers notwithstanding this and other serious obstacles implies that they offer great advantages in other directions, but a clear insight into their history makes it necessary to consider also their disadvantages. It is difficult to understand their slow introduction in any other way.

The term algebra has been frequently used with the meaning that it includes arithmetic, and the term analysis has been used with the meaning that it includes algebra. When a distinction is made with respect to the meanings of these terms it is commonly assumed that algebra is more general than arithmetic since it at first concerned itself with a more general system of numbers than arithmetic does. Similarly, the term analysis is commonly used with a more general meaning than the term algebra since it deals largely with a wider range of concepts than algebra does. These are necessarily only general observa-

tions since there is no commonly accepted line of demarcation between these subjects, as was noted in §3 (volume 13, page 276, 1939), of this journal.

Among the primary facts of the history of mathematics, generalizations and abstractions may be regarded as topmost as regards extensive use and significance. In many cases writers cited special cases when they were aware that the theorems in question were not restricted to these special cases. Generalizations are thus very frequently imposed on the readers who may profit by them in accord with their own insight. The proof of a special case frequently suggests lines of thought which apply to a much more general case and which are more interesting to the reader because they were not explicitly stated as regards details. The mathematical attainments of an individual are largely inspired by re-discoveries due to his own efforts and are more fully comprehended as a result of the insight resulting from these rediscoveries. Only the main features of such general views are commonly embodied in the writings prepared for publications or for an author's individual use. At the close of his *La Géométrie* (1637) R. Descartes stated, "I hope that posterity will judge me kindly, not only as to the things which I have explained, but also as to those which I have intentionally omitted so as to leave to others the pleasure of discovery." This is an important example for teachers.

The slow growth of generalizations is exhibited by the fact that it took more than a thousand years from the time that coordinates were first used until the idea of two coordinate axes was fully established. Even R. Descartes (1596-1650) failed to understand this subject as it is now taught in our schools so that the term "Cartesian system of coordinates" is really a *misnomer*. R. Descartes did not employ any system of coordinates which had not been used earlier by others and some of his remarks show that he was far from understanding even the modern rectangular system of coordinates.* A large number of misstatements relating to the mathematical contributions of R. Descartes appear in the literature of our subject. The greater the credits due to a man the more likely it is that undue credit has also been often given to him, but the mathematical historian naturally is mainly interested in due credit and its proper bestowal.

**Encyclopédie des Sciences Mathématiques*, tome 1, Vol. 3 (1911) p. 51. Cf. G. A. Miller, *Collected Works*, Vol. 2, p. 511 (1938). The closing article of this volume (36 pages) is devoted to the history of mathematics.

The Teacher's Department

Edited by
JOSEPH SEIDLIN and JAMES MCGIFFERT

Visual Aids in Teaching Geometry

By JESSIE MAY HOAG
Jennings, Louisiana

Of the various reasons for the use of visual material in teaching geometry, probably the most obvious is to give meaning to mathematical abstractions. In my mathematics notebook is a prized cartoon clipped from *Collier's* several years ago. It pictures a harassed professor and a broad-shouldered, muscular letter-man standing in front of an equation-cluttered blackboard. The athlete is making an obviously unsuccessful attempt to think, and the dubious but hopeful professor is saying, "All right, take it this way, Mr. Lathum. If you had three footballs and you lost one, how many footballs would you have?"

I remember reading several years ago in THE MATHEMATICS NEWS LETTER* an article in which the author described his experience trying to teach geometry to a young fellow who had failed algebra three times and had finally been given up as hopeless. They started with the proposition which states the congruence of two triangles having two sides and the included angle of one equal respectively to two sides and the included angle of the other. The professor explained it in various ways without success until, finding that the boy had grown up on a farm, he drew rail fences on the blackboard and said, "Now, Mr. Sprague, you and I will fasten these corners together, brace this pen well, then jack it up and move it over on top of the other one." After a little more of such explanation the light dawned, the young man understood the proof, and from then on did satisfactory work in mathematics. Devices used for explanatory purposes are not all as successful as that, but they are usually just as simple. The following examples of materials commonly used for this purpose illustrate a group of devices whose simplicity is in direct contrast to their importance.

*Glenn James, *The Origin of a Mathematical Concept*, Mathematics News Letter, February, 1933.

Colored chalk is an invaluable aid in helping the pupil visualize correspondence in congruent or similar figures. Each pair of corresponding sides may be drawn in the same color, or, when parts of the congruent triangles overlap, one triangle can be shaded one color and the second another color. Another helpful use is in proving congruence by showing that triangles can be made to coincide. One figure may be drawn in one color and the second with another, and as it is proved that each part of one can be made to coincide with a corresponding part of the second the color of the first is drawn over that part of the second. In a complicated construction, coloring just one small part of the figure may serve to emphasize a property that might otherwise be lost among the other relationships. One can underline the hypothesis with one color and the conclusion with another. If it occasionally seems wise to emphasize corrections, they may be made with colored chalk.

One of the most helpful devices I have discovered for helping children understand various facts, especially those about coincidence, is a little collection of colored cardboard polygons and circles. At present the collection contains: congruent triangles; similar triangles, a pair having two sides of one equal respectively to two sides of the other but the included angles unequal, another pair that has two sides and one angle equal to the corresponding parts of another triangle without making the triangles congruent; one equilateral triangle which has been cut along one altitude; a large pair of circles and one smaller pair, each with a sector cut loose from one circle and central angles of various sizes drawn on the other; and numerous regular polygons having from three to eight sides. There are also a large trapezoid and a triangle that can be held together in different positions to form a rectangle or a parallelogram. When a student claims that because two sides of a triangle are equal to two sides of another triangle both pairs of sides can be made to coincide simultaneously, out come the two triangles having two sides of one equal to two sides of the other but having the included angles unequal, and we have a discussion of the meaning of "can be made." The convenience comes in having the proper triangles with labels on the back ready in a box which says, "green triangles congruent, yellow triangles similar, red triangles two sides equal," etc. I cannot begin to list the difficulties cleared away by their use nor can I ever predict from day to day when their quick availability will help answer some new and quite unforeseen question.

These are in the same class with quickly grasped pencils or sticks to illustrate or to help pupils discover the functional relationship between the third side and the angle opposite it in a triangle having

two sides of constant length. Or to show that the angles of a triangle are determined when the lengths of the sides are determined, while those of the parallelogram are not.

Four rulers bolted together to form a loose-jointed parallelogram were constructed originally to facilitate board construction of parallel lines. They have been much more useful in helping pupils to discover and visualize properties of parallelograms and rectangles, and various functional relationships.

As soon as a class has studied congruent triangles, the text books begin to introduce exercises showing how the distance across a river or pond could be measured by applying the facts studied. These are supposed to add concreteness to the work. To make sure that it does, it might be worth while to take the class out doors some sunny day, furnish them sticks and a tape, and give them some such problem as this: After having put up two sticks a few feet apart, tell them to imagine a pond between them. Ask them how they could find the distance across it. If certain points are to be in line, they must determine how to make them so, if distances or angles must be equal, they plan how to make them that way. Members of the group criticize the correctness of suggestions made and discuss the relative ease of various methods. I believe that a method such as this, which involves the planning and materializing of the hypotheses, gives exercises and figures studied later more meaning.

A worth-while exercise for a class studying scale drawings is to reduce the solar system to scale. The planets may be cut from paper, or, if possible, represented by seeds, marbles, balls, apples, or various objects that are the right size. Of course their distances from the sun will be relatively large but they can also be translated into familiar distances such as "the distance from here to the office," "from here to the drive," or "from here to the town of Roanoke." Picture the sun as a blazing ball filling a stated portion of the room and the little ball "planets" revolving about it at about the distances of these familiar places.

Some models function chiefly not so much in explanation as in motivation of the ordinary theorem. It is more interesting to be shown a cardboard model of an old Egyptian level* (See figure 1) and to be asked to discover how it must be constructed to make the base level when the plumb hangs over the base mark, and then to prove it right, than to prove: "The line from the vertex of an isosceles triangle to the mid-point of the base is perpendicular to the base."

*Wells, W. and Hart, W. W., *Modern Plane Geometry*, page 273.
Sanford, V., *A Short History of Mathematics*, page 230.

Notice that the first method, while it connects the fact with a concrete situation, requires just as abstract proof as the second method, and requires even more ability to recognize geometric relationships. Nevertheless, the average pupil feels that it is less abstract and more obvious.

Fig. 1. - Egyptian Level

Fig. 2. - Angle Mirror

An angle mirror* made by a student is hardly accurate enough for very practical use but it makes an interesting problem in construction, and its demonstration gives excuse for proving a fine problem about angles: "Since the angle of incidence equals the angle of reflection, the angle x is twice the angle between the mirrors. (See figure 2.)

Explanation of a pantograph calls for recognition of many geometric principles. A demonstration of how it is used to copy maps or pictures may lead to proof of the proposition:† "If in any parallelogram the four sides, produced if necessary, are cut by a straight line, the four points thus determined will continue to be in a straight line, no matter how much the angles are changed. Furthermore, the distances between these points will remain in a constant ratio to each other. Or the class may work back from the fact that the ratio of

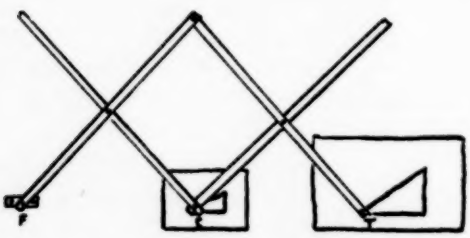


Fig. 3. - Pantograph

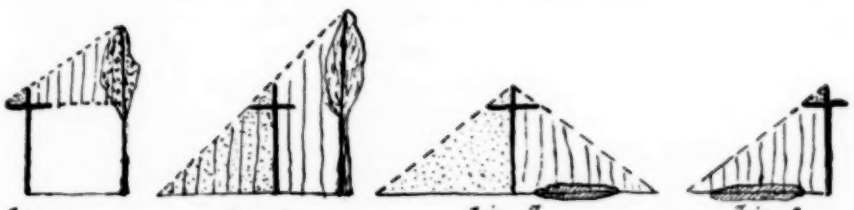
Fig. 4. - Cross Staff

*Shuster, Carl N. and Bedford, Fred L., *Field Work in Mathematics*, pages 39 and 139. If this book is not in your high school library, you should purchase it at once. It is an almost indispensable reference book for high school work.

†Shuster, Carl N. and Bedford, Fred L., *Field Work in Mathematics*, page 137.

distances from the fixed point F to the tracing point T and to the copying point C respectively remains a constant. If this depends on the fact certain triangles are similar, what makes them so? What makes angles of the triangles equal? Finally, how must the instrument be constructed or "set" to make it work—which lengths must be equal and which ones must be proportional?

A copy of the cross-staff* used during the Middle Ages presents other possibilities for encouraging students to discover applications of facts about similar or congruent triangles and then to prove them. Our instrument consists of a piece of material one inch square and five feet long, with a cross stick made of slightly larger material. The cross piece has a hole through the center and two braces to hold it firmly to the staff. (See figure 4.) The methods of measuring inaccessible distances which various classes evolve will differ slightly. In general, however, they will consist of some simple method of sighting an inaccessible point in line with the top and one arm of the cross staff, sometimes of sighting another accessible point, of measuring certain lengths, and of finding the distance of the inaccessible object through the measures of a similar or congruent triangle. Figures 5-8 may suggest some possible relations of the cross-staff and the inaccessible distance to the two triangles. It is also interesting to let pupils determine the easiest way of making a certain measure. They often discover that the method that seems easiest theoretically is the clumsiest practically. For instance, practically it is easier to measure the parts of a cross-staff and find the distance by solving a proportion than to find it directly by measuring as in figure 7. They might also compare methods to determine how they compare in probable accuracy.



Cross Staff

This leads us directly to the third type of visual device—what we might call practical applications of the principles of geometry.

*Shuster, Carl N. and Bedford, Fred L., *Field Work in Mathematics*, page 5.
Sanford, V., *A Short Story of Mathematics*, page 244 for a history of the instrument.

It is to be noticed that this designation of practical is, in some cases, used more with the meaning of "being manifested in action as opposed to speculation or theory" rather than "useful action." We have a very good transit* made by a student which makes it possible to carry out various projects in surveying each year. We may measure a distance across a little gulley in the school yard or the height of the flag pole or the stadium. The pupils choose their own project, elect a foreman, plan their own methods. It is a good way for them to see how the facts of proportion and trigonometry are put to practical use. We have also used this instrument to illustrate the method of laying out levees in the rice fields and of finding slopes in drainage projects.

Occasionally we have made trips to near-by construction jobs to see what methods were being used to make the house level, to make square corners, and to brace it.

Another trip was the result of a project with a slide rule. A pupil who had made a slide rule in a second year algebra class demonstrated its use in solving proportion. The class became interested and about half of them constructed slide rules. These can be made in an hour at a cost of about a cent apiece by cutting and pasting together three heavy strips of card board. Figure 9 gives an end view to show how the paper is cut. The white parts are pasted together and the two shaded parts are pasted to each other. Two strips of semi-logarithmic paper, two-cycle, are pasted on the top layer to form the A and B scales and two other strips of one-cycle semi-logarithmic paper form the C and D scales.† Later we took a trip to a filling station to see how the slide rule was applied in a meter for recording the price of the how the slide rule principle was applied in a meter for recording the price of the gasoline sold. We also went to a dry goods store and studied a Calayard, a slide rule adapted especially to measuring in yards and fractions of yards, and figuring costs of fractions of yards.

Several students in my classes have made sun-dials. One of the best of these was made by a boy who had decided to make a sun-dial before he started geometry. He asked some questions in class one day, and before we got through, we had become involved in a number of things. The pointer for the sun-dial should be set at an angle equal to the latitude of the place where it is to be used, and the computation of the hour angles involves a trigonometric equation involving the latitude.‡ Using the diagram given in figure 10, the class proved that the

*Engle, T. L., *Constructing a Transit as a Project in Geometry*, Mathematics Teacher, November, 1931.

†Bush, William E., *A Student-made Slide Rule*, The Mathematics Teacher, December, 1934, page 405. Paper can be purchased from Keuffel and Esser Co., 127 Fulton Street, New York City.

‡Hogben, Lancelot, *Mathematics for the Millions*, pages 351, 378. This book contains good reference work on sun-dials, latitude, longitude.

latitude angle was equal approximately to the angle of elevation of the pole star. Measuring that angle was a simple project carried out with no more complicated instrument than a protractor with a plumb line attached. (See figure 11.)

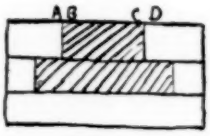


Fig. 9. - Slide Rule

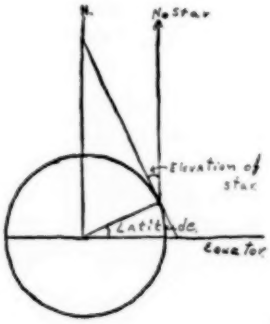


Fig. 10. - Latitude

Fig. 11 - Instrument to Measure Angle of Elevation

Discussion of sun-dial time and Central Standard Time brought the startling question of what time of day was noon in Jennings. After some reference work the following procedure was worked out. A stick was placed in the sun and the end of its shadow marked along the ground at various times during the day. A circle with the pole as center intersected this at two points. The angle formed by lines from these points to the pole was bisected, which gave a north-south line. The time of day when the shadow of the pole coincided with this line was to be counted noon for this point. As near as we could tell, this was eight minutes after twelve, Central Standard Time. Use of radios disclosed the fact that Central Standard Time differed from time in England by six hours. Using our location as six hours, eight minutes west of Greenwich, we computed our longitude. Both latitude and longitude were computed to within a degree, which was lucky, considering the crudeness of our methods and instruments.

Apparently the use of mathematical instruments and various other visual devices has these values: It makes the course more interesting. It is a basis for understanding to the pupil who has not an abstract mind. It gives the pupil who likes to construct things a chance to express himself. It may be practical for the student who later studies engineering or some other application of the facts of geometry, and it appeals to other students who like to see the practical material application of every fact they learn. It takes very little extra time and often saves time because it is easier to prove a fact when one understands it and can visualize it and is interested. And since it adds variety to the course, it acts as a protection from possible boredom.

*Hogben, Lancelot, *Mathematics for the Millions*, page 51.

Mathematical World News

Edited by
L. J. ADAMS

The forty-sixth annual meeting of the American Mathematical Society will be held at Columbus, Ohio, Tuesday to Friday, December 26-29, 1939, in conjunction with the meetings of the American Association for the Advancement of Science, the Mathematical Association of America and the National Council of Teachers of Mathematics. The sessions of the Society will begin Tuesday afternoon and continue through Friday morning. The Mathematical Association will hold its sessions Friday afternoon and Saturday morning, and the National Council will hold sessions Friday morning, and afternoon. All sessions will be at Ohio State University. Some of the addresses scheduled for these meetings are:

1. *The beginnings of mathematical geophysics in Great Britain.* W. D. Lambert, U. S. Coast and Geodetic Survey.
2. *Use of mathematics in the delineation of magnetic and electric anomalies.* Professor Lachlan Gilchrist, University of Toronto.
3. *Gravimetric and seismic methods in exploratory geophysics.* Doctor M. N. Slotnick, Humble Oil and Refining Company.
4. *Mathematical problems in seismology.* Doctor Archie Blake, U. S. Coast and Geodetic Survey.
5. *Address.* Doctor Karl Bigelow, Director of Commission on Teacher Education, American Council on Education.
6. *Discussion.* Professor B. W. Jones, Cornell University; Professor H. H. Downing, University of Kentucky; Professor W. O. Shriner, Indiana State Teachers College; and Professor E. G. Olds, Carnegie Institute of Technology.
7. *Mathematical Reviews.* Doctor T. C. Fry, Bell Telephone Laboratories, and Professor W. B. Carver, Cornell University.
8. *On the technique of generalization.* Professor Henry Blumberg, Ohio State University.
9. *Modular Fields.* Professor Saunders MacLane, Harvard University.
10. *Interior transformations.* Professor G. T. Whyburn, University of Virginia.

11. *Space analogues of function-theoretic results.* Doctor E. F. Beckenbach, The Rice Institute.

The National Council of Teachers of Mathematics will meet Friday, December 29, in Derby Hall, Ohio State University. One session will be devoted to *Training teachers for relational thinking*, with Professor F. L. Wren presiding, and the other will consider *Relational thinking in secondary mathematics as viewed by the college teacher*, with Professor R. L. Morton presiding. The Council will hold a session on Friday afternoon at two o'clock on *Teaching children to do relational thinking*, with Harold Fawcett presiding.

The American Mathematical Society was scheduled to meet at Pasadena, California on December 2, 1939. By invitation of the Program Committee Professor Charles B. Morrey, Jr., of the University of California was scheduled to give an address on *Existence and differentiability theorems for the solutions of variational problems for multiple integrals.*

The American Mathematical Society will meet in New York City on February 24, 1940. By invitation of the Program Committee, Professor B. O. Koopman of Columbia University will deliver an address entitled *The bases of probability.*

A feature offered by *Mathematical Reviews*, the new publication sponsored by the American Mathematical Society, the Mathematical Association of America and Brown University, is its microfilm and photoprint service. By means of this service any article in current mathematical literature is available, at a nominal cost, to the subscribers to *Mathematical Reviews.*

The study of geometry will aid in the development of open mindedness in the student since one of the first things taught him is that he cannot depend on appearance—two triangles are not equal because they look equal) lines are not perpendicular because they appear so. He is taught that all preconceived ideas must be laid aside; that one's opinion is of no value unless it is supported by evidence, that here, the student has a perfect right to say "Submit your evidence." He is taught also that even the author of the text may make mistakes and that the student's word bears just as much weight as that of the mathematician when he has the evidence. What could be better training than this for our students if we desire them to be able to meet every problem of life with unprejudiced minds.—From Arthur J. Lewis' *Contributions of Geometry to the Goals of Education* as published in Mathematics Bulletin, Vol. III, No. 1.

Problem Department

Edited by
ROBERT C. YATES and EMORY P. STARKE

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscript be typewritten with double spacing. Send all communications to ROBERT C. YATES, Mathematics, University, Louisiana.

SOLUTIONS

Late solutions of No. 267, 269, 270, 271 have been received from *J. Barinaga*, University of Madrid.

No. 226. Proposed by *V. Thébault*, Le Mans, France.

Given four spheres, show that the lines joining the internal center of similitude of each pair of spheres with the other meet in a point.

The solution of this problem has already appeared, this Magazine, October, 1939, pp. 52-53. The following remarks are offered by *N. A. Court*, University of Oklahoma.

Given the four spheres (A), (B), (C), (D), let X, X'; Y, Y'; Z, Z'; U, U'; V, V'; W, W' be the centers of similitude of the pairs of spheres (B), (C); (C), (A); (A), (B); (D), (A); (D), (B); (D), (C) respectively. Since the centers of similitude of two spheres are harmonically separated by the centers of the spheres, and the six external centers of similitude X, Y, Z, U, V, W lie in a plane, say, (P), the twelve centers of similtitude considered form a desmic system.*

It follows immediately that the lines $X'U'$, $Y'V'$, $Z'W'$ joining the pairs of internal centers of similitude pass through the harmonic pole M of the plane (P) with respect to the tetrahedron $ABCD$ having for vertices the centers of the given spheres.

*Nathan Altshiller-Court, *Modern Pure Solid Geometry*, p. 240, art. 735. The Macmillan Company, 1935.

Furthermore, the three triads of lines $U'X'$, VY, WZ; $V'Y'$, UX, WZ; $W'Z'$, UX, VY, have each a point in common, namely, the harmonic conjugates M', M'', M''' of the point M for the pairs of points U', X'; V', Y', W', Z'; respectively.*

No. 245. Proposed by *V. Thébault*, Le Mans, France.

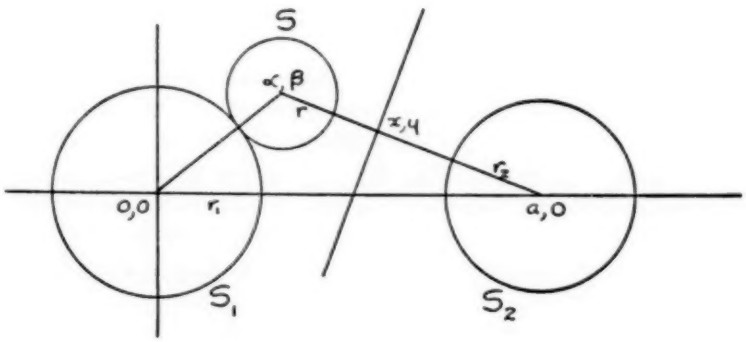
A sphere S of given radius rolls upon a fixed sphere S_1.

(1) The center of similitude of S and another fixed sphere S_2 describes a sphere.

(2) The radical plane of the spheres S and S_2 envelopes a quadric.

(3) Examine the same questions if the sphere S rolls upon two fixed spheres S_1 and S_1'.

Solution by *C. W. Trigg*, Los Angeles City College.



Consider two fixed circles with centers $(0,0)$ and $(a,0)$ and radii r_1 and r_2, respectively. On the first of these circles there rolls a circle of radius r and variable center (α,β). Set $r_1+r=l=\sqrt{\alpha^2+\beta^2}$. The internal center of similitude (x,y) of two circles divides the line of centers into segments proportional to the radii of the circles. Hence

$$\frac{x-\alpha}{a-\alpha} = \frac{r}{r+r_2} = \frac{y-\beta}{-\beta}.$$

So $x(r+r_2)=ar+\alpha r_2$ and $y(r+r_2)=\beta r_2=r_2\sqrt{l^2-\alpha^2}$. When the parameter, α, is eliminated between these two equations we have,

$$[x(r+r_2)-ar]^2+y^2(r+r_2)^2=r_2^2l^2.$$

*ibid., p. 233, art. 715, table B.

Hence the locus of the center of similitude of the two circles is a circle centered on the line of centers of the two fixed circles.

The slope of the line of centers of the variable circle and the second fixed circle is $-\beta/(a-\alpha)$. Hence the equation of the radical axis of these two circles, which passes through the internal center of similitude, is

$$y-r_2\beta/(r+r_2)=(a-\alpha)[x-(ar+\alpha r_2)/(r+r_2)]/\beta.$$

This may be written in the form,

$$L=y(r+r_2)\sqrt{l^2-\alpha^2}-r_2(l^2-\alpha^2)-(a-\alpha)[x(r+r_2)-ar-\alpha r_2]=0.$$

$$\partial L/\partial\alpha=\frac{-\alpha y(r+r_2)}{\sqrt{l^2-\alpha^2}}+ar_2+x(r+r_2)-ar=0.$$

When the parameter α is eliminated between these two equations we have:

$$l^2(r+r_2)^2y^2+[x(r+r_2)-ar]^2(l^2-a^2)-r_2^2l^2)(l^2-a^2)=0.$$

That is, the radical axis envelopes a conic which is symmetrical to the line of centers of the two fixed circles.

Now consider spheres with the same radii and centers as these circles. The internal center of similitude of the variable and the second fixed sphere coincides with that of the circles, and the radical plane of the spheres is perpendicular to the line of centers and passes through the center of similitude. Hence, if the plane configuration be revolved about the line of centers of the fixed circles, we have

(1) the center of similitude of S and S_2 describes a sphere centered on the line of centers of S_1 and S_2; and

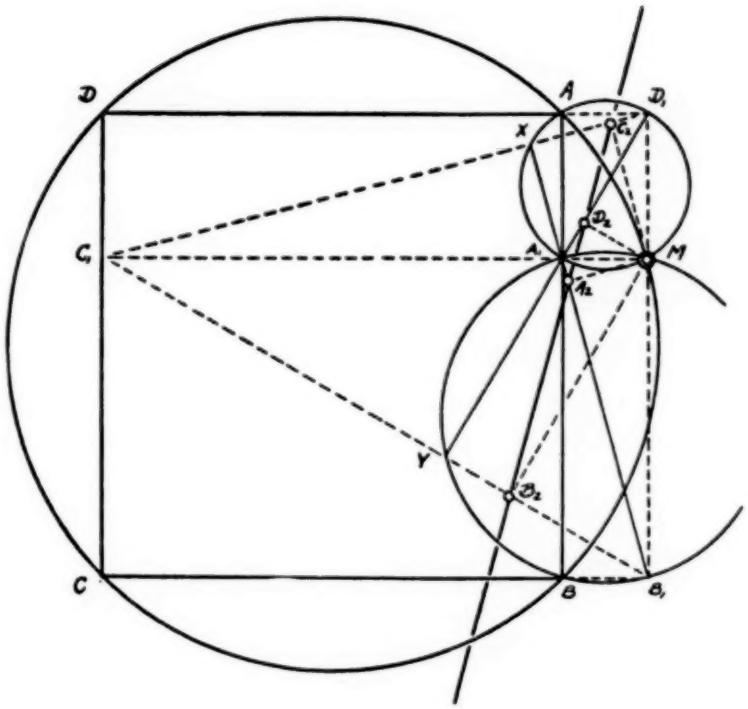
(2) the radical plane of S and S_2 envelopes a quadric surface which is symmetrical to the aforesaid line of centers.

(3) If S rolls upon two fixed spheres S_1 and S_1', then its center describes a circle and the center of similitude of S and S_2 describes a circle which lies on sphere (1). Clearly, the radical plane of S and S_2 envelopes a quadric cone, symmetrical to the line of centers of S_1 and S_2, with its vertex on that line, and which meets the surface (2) in a circle.

No. 286. Proposed by *V. Thébault*, Le Mans, France.

Let M be an arbitrary point of the circle circumscribing the square $ABCD$. Let A_1, B_1, C_1, D_1 be the orthogonal projections of M upon AB, BC, CD, DA; and A_2, B_2, C_2, D_2 the projections of M upon A_1B_1, B_1C_1, C_1D_1, D_1A_1. Show that A_2, B_2, C_2, D_2 are collinear.

Solution by *P. D. Thomas*, Norman, Oklahoma.



We make use of the following theorem: The feet of the three perpendiculars dropped upon the sides of a triangle from a point of its circumcircle are collinear.*

If A_1B_1 is extended to meet C_1D_1 in X, and A_1D_1 extended to meet B_1C_1 in Y, then it is clear that angles B_1XD_1 and B_1YD_1 are both right angles. The points X, D_1, M, A_1 lie on a circle whose diameter is A_1D_1 since A_1XD_1 and A_1MD_1 are right triangles with common hypotenuse A_1D_1. Similarly, the points A_1MB_1Y lie on a circle upon A_1B_1 as diameter.

Hence the points A_2, C_2, D_2 are collinear by the theorem stated above. Also the points A_2, B_2, D_2 are collinear by the theorem. But these two lines have two points, A_2 and D_2, in common so the two Simpson lines are coincident; that is, the points A_2, B_2, C_2, D_2 are collinear.

Also solved by *Walter B. Clarke, Albert Farnell, C. W. Trigg,* and the *Proposer.*

*The Simpson Line, see Altshiller-Court, *College Geometry*, p. 115.

No. 288. Proposed by *L. E. Bush*, College of St. Thomas, Minnesota.

Part I. Let D and A' be any two points on side AC of triangle ABC and E any point on BD. Let CE intersect AB at M, AE intersect BC at N, DN intersect $A'B$ at M', and MM' intersect AC at D'. Prove that D and D' separate A and A' harmonically.

Part II. Use the above theorem to prove that if in the triangle ABC, D is the foot of the perpendicular from B to AC, and if M and N are points on AB and CB respectively, such that AN and CM intersect on BD, then BD bisects angle MDN.

Solution by *Walter B. Clarke*, San Jose, California.

Part I. If we let DN cut AB at H then (AB,MH) is a harmonic range and $M'(AB,MH)$ is a harmonic pencil. Whence, on transversal AC, (AA',DD') is a harmonic range. (See Altshiller-Court, College Geometry, Sec. 261).

Part II. Let MN cut AC at E and BD at F. Then (AC,DE) is a harmonic range and $B(AC,DE)$ is a harmonic pencil. (MN,EF) is a harmonic range and since BD is perpendicular to DE, then FD bisects angle MDN. (See *ibid.* Sec. 263)

Also solved by *L. M. Kelly*, *C. W. Trigg*, and the *Proposer*.

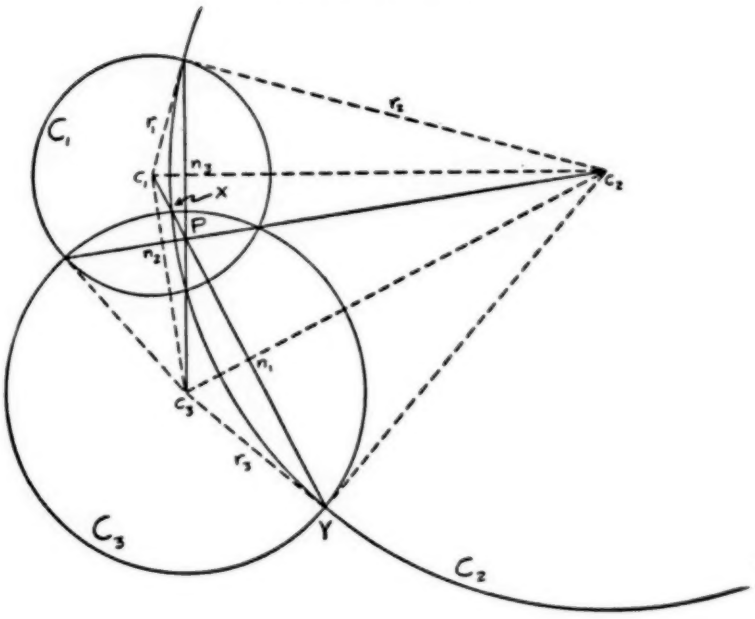
No. 290. Proposed by *J. W. Peters*, University of Illinois.

Given three mutually orthogonal circles C_1, C_2, C_3 in a plane, with centers c_1, c_2, c_3. The point n_3 is the inverse of c_1 in C_2 and the inverse of c_2 in C_1; n_2 is the inverse of c_1 in C_3 and the inverse of c_3 in C_1; n_1 the inverse of c_2 in C_3 and c_3 in C_2. If P is the radical center of C_1, C_2, C_3, prove that P is the inverse of n_1 in C_1, the inverse of n_2 in C_2, and the inverse of n_3 in C_3.

Solution by *L. M. Kelly*, Lawrence, Massachusetts, figure supplied by *C. W. Trigg*, Los Angeles City College.

Because of the orthogonality condition, the common chord of C_1 and C_3 is the polar of c_1 with respect to the circle C_3. Likewise XY, the common chord of C_2 and C_3, is the polar of c_2 with respect to the circle C_3. Therefore, P is the pole of c_1c_2 and thus n_3 is the inverse of P with respect to C_3. Similar remarks can be made for the other circles.

Also solved by *D. L. MacKay*, *Johannes Mahrenholz*, *C. W. Trigg*, and the *Proposer*.

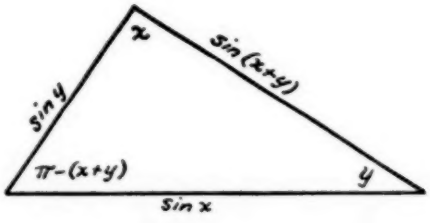


No. 292. Proposed by *Daniel Arany*, Budapest, Hungary.

Establish the following identity:

$$\sin^2 x = \sin^2 y + \sin^2(x+y) - 2 \sin y \sin(x+y)\cos x.$$

Solution by *Johannes Mahrenholz*, Cottbus, Germany.



Consider x and y as angles of a triangle with circumdiameter equal to unity. Then by the law of cosines the identity becomes obvious. The restrictive character of the angles is not an essential obstacle.

Also solved by *Leo F. Epstein, Albert Farnell, Virginia Felder, James K. Hitt, L. M. Kelly, Annie Lyles Lang, D. L. MacKay, L. L. Merrill, L. Nowetcheski, Charles Templeton*, and *C. W. Trigg*.

No. 300. Proposed by *G. W. Wishard*, Norwood, Ohio.

Prove that any odd square except 1 may be formed as the sum of two or three squares of positive integers.

Solution by *D. L. MacKay*, Evander Childs High School, New York.

We employ the following known facts:

(1) Every odd prime is of one of the forms $4k \pm 1$.
(2) Every prime of the form $4k+1$ has a unique representation as the sum of two squares.
(3) The sum of two squares has no non-square factor of the form $4k-1$.
(4) Every integer can be expressed as the sum of four squares (which may include zero).

The odd number $N(>1)$ is divisible by some odd prime p, thus $N=pr$, $r \geqq 1$. If p is of the form $4k+1$, by (2) we put $p=a^2+b^2$, then N^2 is a sum of two squares as follows:

$$N^2=p^2r^2=(a^2+b^2)^2r^2=(a^2-b^2)^2r^2+(2ab)^2r^2. \tag{5}$$

If on the other hand p is of the form $4k-1$, we have by (4) and (3), $p=a^2+b^2+c^2+d^2$, where not more than one of a, b, c, d can be zero. Then N^2 is a sum of three squares as follows:

$$\begin{aligned} N^2=p^2r^2&=(a^2+b^2+c^2+d^2)^2r^2 \\ &=(a^2-b^2-c^2+d^2)^2r^2+4(ab-cd)^2r^2+4(ac+bd)^2r^2. \end{aligned} \tag{6}$$

It is easy to show that these three squares are all different from zero:

$$a^2-b^2-c^2+d^2=0 \quad \text{or} \quad a^2+d^2=b^2+c^2$$

implies $$p=2(a^2+d^2)=(a+d)^2+(a-d)^2$$

contrary to (3);
$ab-cd=0$ implies $p=a^2+2ab+b^2+c^2-2cd+d^2$ contrary to (3); and the like for $ac+bd=0$. Thus the proposed statement is established.

Further discussion: (2), (3) and (5) provide necessary and sufficient conditions that N^2 be a sum of two squares. It may be shown that every odd square, except 1^2 and 5^2, is a sum of three squares. In (6) this result is already shown except for values of N whose prime factors are all of the form $4k+1$. Suppose N has a prime factor $p \neq 5$ of the form $4k+1$. Then $p=a^2+b^2$ by (2) and $p^2=(a^2-b^2)^2+(2ab)^2$. If a or b is a multiple of 5, we may put $2ab=5s$ and then have

$$N^2=p^2r^2=(a^2-b^2)^2r^2+(3s)^2r^2+(4s)^2r^2.$$

If neither a nor b is divisible by 5, their squares are of the forms $5k \pm 1$. If now a^2 and b^2 have the same form, then $a^2 - b^2$ is a multiple of 5, say $5t$, and we have

$$N^2 = p^2r^2 = 9r^2t^2 + 16r^2t^2 + (2ab)^2r^2.$$

If a^2 and b^2 have different forms then $a^2 + b^2$ is a multiple of 5 contrary to the hypothesis on p. Thus N^2 is a sum of three squares unless its prime factors are all 5. Finally let $N = 5^m$, $m > 1$. Then

$$N^2 = 5^{2m} = 5^{2m-4}(15^2 + 12^2 + 16^2)$$

which completes the proof.

EDITOR'S NOTE: Many of the above ideas were discussed by Legendre (1798). Catalan, *Mathesis* (2) 1894, proved that if p is not a sum of two squares, then p^2 is a sum of three squares. The four-square theorem, (4) above, was proved by Lagrange in 1770. For an exhaustive account of the history of related problems, see Dickson, *History of the Theory of Numbers*, v. 2, ch. 8.

DEAR SIR:

The day after the October number of the NATIONAL MATHEMATICS MAGAZINE was received, I submitted the excerpt from the Fourth Canto of the Song of Hiawatha as it appeared on page 59 to my freshman physics class. In this class we use the text, *Mechanics, Molecular Physics, Heat and Sound*, by Millikan, Roller and Watson. The question as proposed to them was modified to read, "If the ten arrows were shot at one second intervals, which performance was the limit of Hiawatha's ability, and if the arrow which he raced was shot at maximum range, how fast could Hiawatha run?"

This seems a more reasonable interpretation than Wright's, for if Hiawatha could keep ten arrows in the air at maximum range, he could keep thirteen in the air by shooting them vertically.

At the next meeting of the class one of my students submitted the attached solution, which merits preservation.

Sincerely yours,
C. W. TRIGG.

HIAWATHA AND THE ARROW

By FRED F. NICHOLSON

As he shot the arrow upward,
To himself thought Hiawatha:
"If I shoot ten arrows upward
With a second 'twixt the arrows,
And the first one hits the greensward
As the last the bow is leaving,
For nine seconds has the first one
Traveled on its aerial journey.
And the time it travels upwards
Equals that it travels downwards
So the time it travels upwards
Equals four and one-half seconds.

Big Chief Millikan has told us
That the speed with which it started
Equals time that it must travel
Multiplied by 'g' a constant,
Gravity's acceleration.
Thus the speed with which it started
Equals four and one-half seconds
Times nine hundred and eighty,
Centimeters second second.
This is equal to a speed of
Forty-four and one-tenth meters
Traveled upwards in a second.

II

"At this speed I'll shoot an arrow
At an angle that will give me
Greatest distance that the arrow
Travels forward when I shoot it.
Forty-five degrees is angle
Since the value sine two alpha
Cannot larger be than one is,
And it equals one when alpha
Is as forty-five degrees is.
Big Chiefs Millikan and Roller,
Not forgetting Big Chief Watson,
Say that horizontal speed is
Equal to cosine of angle
Of propulsion of projectile
Multiplied by speed of arrow.
Forty-five degrees has cosine
Equal to a one divided
By square root of two which equals
Seven hundred seven thousandths.
This time speed of arrow which is
Forty-four and one tenth meters
Equals thirty-one and eight tenths
Meters, I must run per second,
If I do not want the arrow
Falling to the ground beyond me
Or while I am running swiftly
Hitting me upon my ankle."

III

Once again thought Hiawatha:
"Injuns don't use metric system
And they don't use English ditto
And they don't do Physics problems.
Why the —— did I do this one?"

• • • •

Hiawatha, Hiawatha,
He could run in any clime
And departing leave behind him
Footprints on the sands of time.

PROPOSALS

No. 325. Proposed by *Walter B. Clarke*, San Jose, California.

Given three points A, B, C. Let D be the reflection of A in BC. Let BD cut the circumcircle of ABC in E. Produce DC to F so that $DC = CF$. Show that FE is perpendicular to BE.

No. 326. Proposed by *G. W. Wishard*, Norwood, Ohio.

A^2, B^2, and C^2 form an arithmetical progression: find a two-parameter solution for A, B, and C in integers.

No. 327. Proposed by *D. L. MacKay*, Evander Childs High School, New York.

Given the line segments a and b, $a > b$, which are the first two terms of a decreasing geometric progression. Give a geometrical construction for the limit of the sum of the infinite progression.

No. 328. Proposed by *J. Rosenbaum*, Bloomfield, Conn.

Denoting the rth coefficient in the expansion $(a+b)^n$ by (n, r) and assuming that n is a positive integer and that $n>r+k-2$, show that

$$(n, r)=(n+k, r+k)-(k, 2)(n+k-1, r+k)+(k, 3)(n+k-2, r+k)$$
$$-(k, 4)(n+k-3, r+k)+\cdots+(-1)^k(n, r+k).$$

No. 329. Proposed by *Robert C. Yates*, Louisiana State University.

P is a point in the plane of a given circle C. With the compasses alone, find the intersections of the circle and its diameter through P.

No. 330. Proposed by *E. C. Kennedy*, Texas College of Arts and Industries, Kingsville, Texas.

A certain triangle has one side three times another and the included angle is 60°. Find the triangle with integral sides, the largest of which is less than 150, which is the best possible approximation to the given triangle.

No. 331. Proposed by *Daniel Arany*, Budapest, Hungary.

A tangent is drawn to the circumcircle of A,B,C at the point T. Perpendiculars dropped from A,B,C to this tangent meet it in P_1, P_2, P_3 and the circle in R_1, R_2, R_3.

1. The perpendiculars from R_1, R_2, R_3 upon the sides BC, CA, AB are concurrent at M, a point on the circumcircle.
2. The perpendiculars from P_1, P_2, P_3 upon the sides BC, CA, AB are concurrent at a point S (the orthopole of the tangent at T) on the Simson line of the point M (referred to the triangle ABC).
3. The Simson line of the point T passes through S.

Bibliography and Reviews

Edited by
H. A. SIMMONS

Applied Mathematics in Chemical Engineering. By Thomas K. Sherwood and Charles E. Reed. McGraw-Hill Book Company, Inc., New York, 1939. xi+403 pages.

This book is very different from the usual text-book, and requires a rather detailed description. It includes ordinary and partial differential equations, empirical equations, alignment charts, errors and precision of measurement, as well as those topics of advanced calculus which are apt to be of use to the chemical engineer. Proofs of mathematical formulas are omitted; instead, the formulas are illustrated by problems which are set up and worked through in detail.

The text begins with a review of calculus, including graphical and numerical differentiation and integration. Then a chapter is devoted to problems in chemical engineering which lead to ordinary differential equations, illustrating the method of setting up the equations. A good but very condensed outline of the analytical methods of solving ordinary differential equations is followed by examples illustrating graphical and numerical solution by a modified Euler method, and by the methods of Runge-Kutta, Milne, and Picard. Application to higher order differential equations is illustrated briefly.

The next chapter is on partial derivatives, and is particularly useful because it gives a good description of the Jacobian notation for partial derivatives, which allows any desired formula of thermodynamics to be written from a table of one-page size.

A chapter on infinite series includes Fourier Series and Bessel's Functions. The chapter on partial differential equations includes problems leading to Fourier Series and a series of Bessel's Functions. Of particular interest are problems illustrating the Schmidt of graphical or numerical solution of the diffusion equation. There is no problem involving Legendre Polynomials.

A chapter on numerical analysis includes the Newton and the Lagrange interpolation formulas as well as conventional material on empirical equations, including least squares. The chapter on graphical treatment includes problems illustrating the use of logarithmic and and semi-logarithmic paper and other non-uniform scales for special purposes, polar and especially triangular coordinates and alignment

charts. The final chapter on theory of errors is conventional. An appendix contains a collection of 38 problems (without answers) arranged in the same order as the material in the text.

The book is unusual because of its emphasis on graphical and numerical approximate solution of problems, and because of the large variety of problems which are set up and solved. The statement beginning on the bottom of page 29 is in error concerning the number of arbitrary constants in the equation, and the order of the differential equation, of a general parabola. On page 185, it should be stated that the comparison test is for a series of positive terms. In general the book is very precise and accurate.

Whenever the chemical engineer meets a problem of mathematical nature, he will generally find a similar problem solved in this book; it would seem invaluable to him. It has a good index. It can well be used as a text-book by the graduate and gifted senior student in chemistry or chemical engineering, after they have had a course in differential equations.

The mathematics teacher and mathematical physicist will find the book very useful in its problems in numerical and graphical solution of differential equations; in particular, the Schmidt method of solution of the diffusion equation should be more widely known.

Carnegie Institute of Technology. TURNER L. SMITH.

Plane Trigonometry. By Stratton and Daugherty. Prentice-Hall. June, 1939. $2.25.

This text, in eighty pages, offers material for a brief introduction to *Trigonometry*. The average high school or college class should be able to cover the entire book without difficulty.

The text opens with the definition of the trigonometric functions in terms of the right triangle, the eight fundamental identities, and their application to proving identities. The introduction of Greek letters without any explanations, seems rather abrupt, to say the least. Determination of the values of the functions of 0° and 90°, particularly $\tan 90° = \infty$, without any discussion of the limiting process, or the significance of the symbol ∞ strikes one as poor mathematics as well as poor pedagogy.

The early introduction of trigonometric equations is a worthwhile step toward a greater unification of the subject, Why not use inverse functions early and often, as well? By so doing, the student would become familiar with them and avoid the jolt of what seems to many, an extraneous topic at the last.

Chapter 3 (p. 14) introduces the tables of natural functions and applies them to the solution of right triangles. No values are given for secant and cosecant, the authors using the Pythagorean theorem when finding the hypotenuse. The text would be improved by the addition of at least one example of interpolation for a decreasing function, instead of leaving this entirely to the initiative of the student or the instruction by the teacher. No check on results other than the answer book are given, so that the student is not given that complete independence which should be a part of his mathematical training, and which he *must have* if he applies himself to any other than set text-book problems.

For as brief a course as this, the introduction of two distinct notations for negative characteristics of logarithms seems needless, and the use of cologarithms seems somewhat over-emphasized.

In Chapter 5 we find the first mention of general angles. The use of separate formulas for functions of $\pm\theta$, $90° \pm \theta$, $180° \pm \theta$, $270° \pm \theta$, $360° \pm \theta$ is accurate enough, but strikes this reader as an unnecessary addition to the number of formulas to be remembered, in view of the fact that a single formula can be made to replace all of them. It is noticeable that no attempt is made to show the student that the eight fundamental identities hold for *any* angle.

In the four-page chapter on radian measure, the authors made the rather unusual distinction between *angular measure* and *circular measure*, meaning measure in *degrees* and *radians*, respectively. The student is asked to solve a number of conversion problems with a small table up to nine degrees, nine minutes and nine seconds, although the tables include a good conversion table.

There is a good discussion of line values of the functions. One could wish for more specific and accurate language in discussing graphs. For instance, there is doubt as to the selection of units "so as to give a well-balanced curve." The "double loop" of a sine-curve is certainly an unusual description of a curve which does not intersect itself.

In the derivation of a formula for $\sin(x+y)$, one learns very casually that certain lines are supposed to be perpendicular. The direct geometric derivation is given only for the sine, the other associated derivations depending on this one from $\cos\theta = \sin(90° - \theta)$ and other identities; $\cot(x \pm y)$ is conspicuous by its absence.

Fourteen pages cover the solution of oblique triangles, with no methods of checking solutions given except a very brief mention of Mollweide's formulas.

The book concludes with a chapter on inverse functions and a summary of 33 of the formulas developed.

A teacher who is primarily and chiefly interested in passing on to the student facts, formulas, and their applications, and who feels that the insistence upon strict accuracy in reasoning is unnecessary or beyond the grasp of his students, might find this text very acceptable.

In the opinion of this reader, the correction of some of the faults noted, together with greater care in leading the student to think his way through to the formulation of rules, would make this a much more valuable textbook.

Marquette University. H. P. PETTIT.

An Introduction to Modern Geometry. By Levi S. Shively. John Wiley & Sons, New York, 1939. xi+167 pages. Price $2.00.

here have been few books, written and published in this country, presenting material either for a course in the preparation of prospective teachers of geometry in our high schools or for the further study of Euclidean geometry by those interested in the subject. Thus ten years have passed since R. A. Johnson published his *Modern Geometry* and fifteen years since the appearance of Altshiller-Court's *College Geometry*.

Unlike both former books the material here can be covered in one semester and hence is truly an introduction to the geometry of the triangle. Like *Modern Geometry* no background beyond elementary trigonometry is necessary; like *College Geometry* numerous problems in construction are given and the fundamental concepts in projective geometry of harmonic points and lines and poles and polars as to a circle are stressed.

However this text goes further in utilizing projective geometry concepts as we find a discussion of projectivity, cross-ratio, and one dimensional involutions.

Chapter II deals with constructions with ruler and compass. Mascheroni's geometry of the compass is given together with a brief study of the division of the circle into equal arcs and Lemoine's criteria for exactness and simplicity of construction. Chapter V is concerned with the geometry of the triangle, and in Chapter XII we find some well-known geometrical theorems. The other chapters deal with similar and homothetic figures, the theorems of Ceva and Menelaus harmonic points and lines, coaxal circles, inversion, poles and polars, cross-ratio, and involution.

More than 350 exercises scattered through the chapters with 100 miscellaneous supplementary exercises at the end of the book enable the student to test his proficiency. The proof reading and illustrations in the text seem excellent.

Western Reserve University. J. R. MUSSELMAN

Medieval Number Symbolism, Its Sources, Meaning and Influence On Thought and Expression. By Vincent Foster Hopper. Columbia University Press, New York, 1938. xii+241 pages. $2.90.

This book, in so far as it is of interest at all, is distinctly of literary, and not of mathematical, interest. But even from the literary point of view, Professor Hopper's presentation of a very fascinating subject is unfortunate. His book has all the ear-marks of an early draft of a Ph.D. Dissertation. It reads like a tiresome transcription of scholarly notes from an imposing collection of index cards: all *staccato* and no *legato*. In fact, we have here the raw materials of a book. That is all.

We learn from the Preface that the author's aim is to explain the relationship of the ancient primitive, astrological and Pythagorean number theories to medieval number-symbolism and to the numerical framework of Dante's *Comedy*. The explanation, however, rarely goes further than juxtaposition. The first three chapters are devoted respectively to the three sources of medieval number-theory; and these have avowedly little or nothing original in them. The fourth chapter describes the transition from the older philosophies to the Christian number theory *via* the Gnostics and the Neo-Pythagoreans. The fifth and sixth chapters plunge us into the subject matter proper, giving a detailed account of the symbolic and theological significance attributed to numbers in the literature of the Middle Ages. The seventh, and by far the best, chapter is devoted to number symbolism in Dante. And finally we are offered a brief Appendix consisting of disconnected remarks on number symbols of Northern Paganism. Like the rest of the book, it is well studied with erudite footnote references.

Professor Hopper's study is valuable as a mine of information. Its usefulness would certainly have been enhanced if the author had ventured to interpret as well as to list the literary data he has so painstakingly unearthed. Its importance to a modern student would have been made more apparent if he had given some indication of the survival of number symbolism in the literature of later periods. In *Paradise Lost* (viii, 419-426), for example, Adam, pleading with the Almighty for a consort, exclaims:

> *"No need that thou*
> *Shouldst propagate, already infinite,*
> *And through all numbers absolute, though One:*
> *But Man by number is to manifest*
> *His single imperfection, and beget*
> *Like of his like, his image multiplied,*
> *In unity defective; which requires*
> *Collateral love, and dearest amity."*

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